New Distribution-Free Tests for Stochastic Dominance

by

Kuan Xu
and
Gordon Fisher & Douglas Wilson
Concordia University

Working Paper No. 95-02

February 1995

DEPARTMENT OF ECONOMICS

DALHOUSIE UNIVERSITY
HALIFAX
NOVA SCOTIA B3H 3J5

New Distribution-Free Tests for Stochastic Dominance

Kuan Xu[1]
Dalhousie University, Halifax, Canada B3H 3J5

Gordon Fisher and Douglas Willson
Concordia University, Montreal, Canada H3G 1M8

First Version: April 1993
Current Version: November 1994

[1]We wish to thank C. M. Beach, A. Shorrocks, P. Talwar, J. Tomberlin, S. Wang, and the seminar participants at Concordia University, Dalhousie University, the University of Western Ontario, the Canadian Economic Association 1994 Annual Meeting, and the Econometrics Society 1994 European Meeting for their comments and suggestions.

Abstract

Existing test procedures for stochastic dominance often misspecify the null hypothesis, and are inappropriate for data characterized by cross-section or weak temporal dependence. This paper develops new distribution-free test procedures for first- and second-degree stochastic dominance under very general conditions. An example considers real U.S. T-bill yields, and illustrates that the tests are straightforward to interpret and apply.

Keywords: stochastic dominance, test-statistics, weak dependence, quantile function estimates, moving block bootstrap

JEL classification: C12, G11, I30

1 Introduction

Tests for stochastic dominance are important tools for the comparison of distributions between pairs of random variables. In finance, dominance criteria can potentially provide an unambiguous ranking of the desirability of different assets while placing only general restrictions on the preferences of investors. In welfare economics, dominance concepts allow for the ranking of income distributions using generally accepted welfare criteria.[1] Existing dominance test procedures suffer from a number of weaknesses.

One weakness concerns restrictions on the class of distribution functions that may be used as a basis for testing. Early literature in this area typically ignored the sampling errors associated with estimates of the empirical distribution or quantile functions [see Levy and Hanoch (1970), Porter, Wart and Ferguson (1973) and Vickson (1977), Vickson and Altman (1977), Levy and Kroll (1979), and Kroll and Levy (1980)]. More recently, this issue has been addressed but only at the cost of restricting the class of parametric distributions discussed [see, for example, Deshpande and Singh (1985), and Stein and Pfaffenberger (1986)]. Since dominance criteria are attractive primarily because they allow for a ranking of returns on risky assets or income distributions, while placing only weak restrictions on preferences, it is important that dominance tests retain a degree of generality, and hence remain nonparametric in nature.

Another weakness of existing dominance tests is that they often specify the null hypothesis improperly [see, for example, Tolley and Pope (1988), Bishop, Formby, and Thistle (1989), and Bishop, Chakraborti, and Thistle (1989).] For example, most test procedures make use of the null hypothesis that two distribution or quantile functions are identical. The hypothesis of dominance may be viewed as an hypothesis of inequality in a particular direction between two distribution or quantile functions. If such an hypothesis is rejected, then dominance cannot be sustained, a result that may or may not be caused by the equality of the two distribution or quantile functions. On the other hand, if the null hypothesis of equality is rejected, then the two distributions cannot be said to be equal, but the cause may or may not be that one dominates the the other. Thus the null hypothesis of equality is not very helpful in providing information about dominance [see Levy (1992), p.574].

A third weakness is that existing test procedures are not appropriate for data that exhibit weak dependence within samples and/or association between samples. This is particularly important in finance applications, where there is substantial evidence suggesting that returns on risky assets are not *i.i.d.*; instead, the emerging empirical consensus suggests that while returns on assets may be unconditionally homogeneous (i.e.

[1]The theory of stochastic dominance analysis was developed by Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Whitmore (1970).

identically distributed), they are conditionally heterogeneous with time varying conditional variances (and possibly higher-order moments as well).[3] Allowing for dependence between the random variables under consideration is also desirable, because returns on assets are obviously determined and sampled jointly, and a positive correlation between returns will reduce the variance of the differences between distribution or quantile function estimates. Similarly, a comparison among income distributions of an economy over time should allow for dependence when the same set of respondents is sampled over time.

This paper develops new distribution-free test procedures for stochastic dominance that address all of these weaknesses. The paper is organized as follows. Section 2 serves to review the basic concepts of stochastic dominance and establish some notation. Section 3 develops the new distribution-free test procedures. In section 4, two special cases of the proposed test-statistics are presented as examples. An empirical example evaluating dominance relationships between real U.S. T-bill yields is presented in section 5. Finally, some concluding remarks are made in section 6.

2 Basic Concepts of Stochastic Dominance

2.1 Notation and Definitions

Stochastic dominance criteria are useful for ranking distributions without placing parametric restrictions on preferences. We will consider dominance criteria associated with two general classes of utility functions:

Definition 1 *Denote two classes of utility functions as U_i for $i = 1, 2$. U_1 includes all the functions u with $u' \geq 0$; and U_2 includes all the functions u with $u' \geq 0$ and $u'' \leq 0$.*

U_1 is a class of monotonically increasing utility functions. This class of functions characterizes a preference for higher utility. U_2 is a class of monotonically increasing and concave utility functions. This class of functions characterizes a preference for higher utility and lower risk.

Definition 2 *X dominates Y in the first-degree and the second-degree, denoted by XD_1Y and XD_2Y, respectively, if*

$$F_Y(w) - F_X(w) \geq 0 \; \forall w \in [a, b] \; (XD_1Y); \tag{1}$$

[3] See Engle (1982) and Baillie and Bollerslev (1990).

and

$$\int_a^w [F_Y(t) - F_X(t)]dt \geq 0 \ \forall w \in [a,b] \ (XD_2Y). \tag{2}$$

Alternatively, strict dominance in the first-degree and second-degree, are defined by (1) and (2) with at least one strictly inequality.

There is a one-to-one correspondence between stochastic dominance in the i-th-degree and a preference ordering based on the expected utility functions of the i-th-class ($i = 1,2$) [see Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), and Bawa (1975)].

Figures 1 and 2 illustrate XD_1Y and XD_2Y using the distribution functions F_X and F_Y. In Figure 1, F_X is to the right of F_Y, i.e., F_X is less than or equal to F_Y for all $w \in [a,b]$. In Figure 2, the sum of the areas created by the two curves from the point of $w = a$ to any $w \in (a,b]$ is always greater than or equal to zero, *i.e.*, $\int_a^w [F_Y(t) - F_X(t)]dt \geq 0$ for all $w \in [a,b]$.

2.2 Decision Rules Based on Quantile Functions

Decision rules for stochastic dominance relations can also be based on quantile functions.

Definition 3 *If a distribution function F is strictly monotonic, the corresponding quantile function of order p, $Q(p)$, is the inverse function of the distribution function F. $Q(p)$ is given by $Pr[w \leq Q(p)] = p$. If the distribution function is weakly monotonic, then* $Q(p) = \inf\{w : F(w) \geq p, 0 \leq p \leq 1\}$.

Theorem 1 *X dominates Y in the first-degree, and the second-degree, denoted by XD_1Y and XD_2Y, respectively, if*

$$Q_X(p) - Q_Y(p) \geq 0 \ \forall p \in [0,1] \ (XD_1Y); \tag{3}$$

and

$$\int_0^p [Q_X(t) - Q_Y(t)]dt \geq 0 \ \forall p \in [0,1] \ (XD_2Y). \tag{4}$$

Alternatively, strict dominance in the first-degree and second-degree are defined by (3) and (4) with at least one strict inequality.

Proof. See Levy and Kroll (1978). □

Figures 3 and 4 show XD_1Y and XD_2Y using the quantiles Q_X and Q_Y. In Figure 3, $Q_X(p)$ is to the left of $Q_Y(p)$, i.e., $Q_X(p) - Q_Y(p) \geq 0$ for all $p \in [0,1]$. In Figure 4, the sum of the areas surrounded by the two curves from the point of $p = 0$ to any $p \in [0,1]$ is always greater than or equal to zero, i.e., $\int_0^p [Q_X(t) - Q_Y(t)]dt \geq 0$ for $p \in [0,1]$.

Dominance criteria and tests of dominance relationships can be based on either distribution or quantile functions and their associated empirical estimates. The choice is primarily a matter of convenience. The tests developed here are based on sample quantiles, although tests based on sample proportions may be even simpler to compute under more restrictive assumptions about the data.[3] Under the general conditions considered in this paper, the tests based on sample quantiles are conceptually simpler because the domain of the quantile functions for both random variables being compared is the interval $[0,1]$, while the domain of the distribution functions are case-specific and may not be the same; this increases the possibility of 'sampling zeros' in tests based on sample proportions.

In financial economics, Definition 1 and Theorem 1 can be applied directly to an analysis of asset choice. In the income distribution literature, the terminology used is a little different from that in financial economics. Consider Generalized Lorenz (GL) dominance as given in Shorrocks (1983) and let F_X and F_Y denote two income distributions. GL curves can be defined over the corresponding quantile functions Q_X and Q_Y, *i.e.*, $GL_X(p) = \int_0^p Q_X(t)dt$ and $GL_Y(p) = \int_0^p Q_Y(t)dt$. GL (*strict*) dominance of an income distribution F_X over another income distribution F_Y is defined as $GL_X(p) \geq GL_Y(p)$ $\forall p \in [0,1]$ (*with a strict inequality for a least one p*). This is equivalent to SSD in Theorem 1.[4]

[3]The degree of computational difficulty depends primarily on the estimation of the variance-covariance matrix of the sample proportions or sample quantiles. Under i.i.d assumptions, estimation of the variance-covariance matrix for a vector of sample proportions is trivial [*e.g.* Anderson (1994)], while variance and covariance estimates for sample quantiles require density estimation. Under the general conditions assumed here, there are no computational advantages to choosing proportions over quantiles.

[4]Shorrocks (1983) shows that the GL dominance is equivalent to preference by all increasing, anonymous, equality-preferring social welfare functions.

3 New Distribution-Free Tests for Stochastic Dominance under General Conditions

As shown previously, the quantile conditions for FSD and SSD are: XD_1Y if and only if $Q_X(p) - Q_Y(p) \geq 0 \ \forall p \in [0,1]$; and XD_2Y if and only if $\int_0^p [Q_X(t) - Q_Y(t)]dt \geq 0 \ \forall p \in [0,1]$. Alternatively, the quantile condition for SSD can be expressed in terms of cumulative quantiles. The cumulative quantile function is given by

$$\Psi(p) = \int_0^p Q(t)dt \tag{5}$$

where $p \in [0,1]$. Thus, XD_2Y if and only if $\Psi_X(p) - \Psi_Y(p) \geq 0 \ \forall p \in [0,1]$.

Decision rules for FSD and SSD have a common structure, and both can be expressed in a more general form. Let θ denote either Q or Ψ. Tests for FSD and SSD are of the form $\theta_Y - \theta_X \geq 0$. Let θ belong to a parameter space Ω, i.e., $\theta \in \Omega$. Let the restricted space R be a subspace of the parameter space Ω, i.e., $R \subset \Omega$. For testing hypotheses such as H_0: $\theta_Y - \theta_X \geq 0$ against H_a: $\theta_Y - \theta_X \ngeq 0$, test-statistics can be designed using both the restricted estimator $\tilde{\theta}_Y - \tilde{\theta}_X$, and the unrestricted estimator $\hat{\theta}_Y - \hat{\theta}_X$. The unrestricted estimator $\hat{\theta}_Y - \hat{\theta}_X$ are in Ω, while the restricted estimator $\tilde{\theta}_Y - \tilde{\theta}_X$ must be in the restricted space R. The tests, therefore, should be able to measure the distance between the restricted and unrestricted estimators (see Theorems 2 and 3 below).

3.1 Assumptions

Denote the two random variables $\{X_t, Y_t\}'$ as the random vector $\{Z_t\}$, where Z_{tj}, $j = X, Y$, or $X_t = Z_{tX}$ and $Y_t = Z_{tY}$. We will assume that $\{Z_t, -\infty < t < \infty\}$ is a stationary ϕ-mixing sequence of random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Thus, if $\mathcal{M}_{-\infty}^k$ and $\mathcal{M}_{k+T}^\infty$ are the σ-fields generated by $\{Z_t, t \leq k\}$ and $\{Z_t, t \geq k+T\}$, respectively, and if $E_1 \in \mathcal{M}_{-\infty}^k$ and $E_2 \in \mathcal{M}_{k+T}^\infty$, then for all $k(-\infty < k < \infty)$ and $T(\geq 1)$,

$$|P(E_2|E_1) - P(E_2)| \leq \phi(T), \quad \phi(T) \geq 0, \tag{6}$$

where $1 \geq \phi(1) \geq \phi(2) \cdots$, and $\lim_{T\to\infty} \phi(T) = 0$.

Assumption 1 *The ϕ-mixing sequence satisfies*

$$\sum_{T=1}^{\infty} [\phi(T)]^{1/2} < \infty. \tag{7}$$

Assumption 1 essentially requires that the dependence between observations dies out as the (temporal) distance between them increases. Mixing assumptions have been employed in a variety of financial market applications to characterize weak dependence. [See Lo (1991), for example.]

Let Z_t have a bivariate density function f_Z, and distribution function F_Z, where $z = (x, y) \in \mathbf{R}^2$. The corresponding marginal density and distribution functions for X and Y are written as f_X and f_Y, and F_X and F_Y, respectively.

The empirical distribution function for j-th variate is given by

$$F_{Tj}(w) = \frac{1}{n}\sum_{t=1}^{T} I(Z_{tj} \leq w), \tag{8}$$

where w is in the support, $j = X, Y$, and $I(A)$ is and indicator function that equals one if the event A holds, and equals zero otherwise. K proportions or probabilities and K corresponding quantiles are chosen for both $\{X_t\}$ and $\{Y_t\}$. $\xi = [\xi^X_{(1)}, \xi^X_{(2)}, \cdots \xi^X_{(K)}, \xi^Y_{(1)}, \xi^Y_{(2)}, \cdots \xi^Y_{(K)}]'$ denotes a $2K \times 1$ vector of quantiles defined in $\mathbf{R}^2$. $P\left\{Z_{tj} \leq \xi^j_{(i)}\right\} = p^j_{(i)}$, where $0 < p^j_{(i)} < 1$, $i = 1, 2, \ldots, K$, and $j = X, Y$. Assume that $p^X_{(i)} = p^Y_{(i)}$ for all i; then $P\left\{Z_{tj} \leq \xi^j_{(i)}\right\} = p^j_{(i)}$ can be written as $P\left\{Z_{tj} \leq \xi^j_{(i)}\right\} = p_i$, *i.e.*, the superscript j of $p^j_{(i)}$ can be suppressed. Thus, the $2K \times 1$ vector, ξ , can be expressed alternatively as

$$\begin{aligned} \xi &= Q_Z(P) \\ &= [Q_X(P), Q_Y(P)]' \\ &= [Q_X(p_1), Q_X(p_2), \cdots Q_X(p_K), Q_Y(p_1), Q_Y(p_2), \cdots Q_Y(p_K)]'. \end{aligned} \tag{9}$$

Assumption 2 *F_Z is strictly monotonic in some neighborhood of ξ in each of its two coordinates and admits of a differentiable continuous density f_Z, such that*

$$0 < f_Z < \infty. \tag{10}$$

Assumption 2 allows for a convenient representation for the limiting variance-covariance matrix of the sample quantiles, and could be relaxed, although at the cost of additional complexity.[5]

If we denote the observations from X and Y $\{x_t\}^T_{t=1}$ and $\{y_t\}^T_{t=1}$, observations can be arranged in increasing order, i.e., $x_{(1)} \leq x_{(2)} \cdots \leq x_{(T)}$ and $y_{(1)} \leq y_{(2)} \cdots \leq y_{(T)}$,

[5] Serfling (1980) discusses this issue in detail.

where $x_{(t)}$ and $y_{(t)}$ are the t-th order statistics of X and Y, respectively. The sample quantiles of order p for X and Y are denoted as $\hat{Q}_{TX}(p) = x_{([Tp]+1)}$ and $\hat{Q}_{TY}(p) = y_{([Tp]+1)}$, respectively, where $[Tp]$ refers to the largest integers that are less than or equal to Tp. For a finite set of quantiles, such as $\hat{Q}_{TX}(P) = \{x_{([Tp_i]+1)}|i = 1, \ldots, K\}$, and $\hat{Q}_{TY}(P) = \{y_{([Tp_i]+1)}|i = 1, \ldots, K\}$, associated with the the abscissae $P = \{p_i|i = 1, \ldots, K\}$, it can be shown that the $2K \times 1$ vector of sample quantiles $\hat{Q}_{TZ}(P) = [\hat{Q}_{TX}(P), \hat{Q}_{TY}(P)]'$ has an asymptotic normal distribution as given in Lemma 1.

Lemma 1 *Under Assumptions 1 and 2, as $T \to \infty$,*

$$\sqrt{T}[\hat{Q}_{TZ}(P) - Q_Z(P)] \xrightarrow{d} N(0, \Lambda), \tag{11}$$

where

$$\Lambda = D^{-1}\mathrm{V}(D')^{-1},$$

$$D = diag[f_X(Q_X(p_1)), \cdots, f_X(Q_X(p_K)), f_Y(Q_Y(p_1)), \cdots f_Y(Q_Y(p_K))]',$$

and

$$\mathrm{V} = \lim_{T\to\infty} E(\mathrm{mm}'),$$

where $\mathrm{m} = \frac{1}{T}[\mathcal{F}_X \mathcal{F}_Y]'$ *with* $\mathcal{F}_j = [(F_{Tj}(Q_j(p_1)) - p_1), \cdots, (F_{Tj}(Q_j(p_K)) - p_K)]$, $j = X, Y$.

Proof. See Sen (1972).□

Lemma 1 is a useful limiting distribution for two sets of quantile points which are estimated for two associated time-dependent stochastic processes. The elements of Λ are fairly complicated and depend on the serial correlation structure of the data. While it is possible to construct consistent estimates of Λ using kernel density estimates and Newey-West style truncation arguments, it is also possible (and much less cumbersome) to employ a bootstrap resampling algorithm to estimate the elements of Λ. In this context, it is important to note that the resampling algorithm must replicate the dependence structure in the data for the bootstrap estimates to be consistent. For stationary ϕ-mixing random variables, the moving-block bootstrap (MBB) developed by Kunsch (1989) and Liu and Singh (1992) will provide consistent estimates of Λ.[6]

As the test-statistics presented below are actually based on differences in sample quantiles, it is sufficient to estimate the variance-covariance matrix of $\hat{Q}_X - \hat{Q}_Y$. Since $Q_X - Q_Y$ can be expressed as HQ_Z, where H is a matrix such that $HQ_Z = Q_X - Q_Y$, the variance-covariance matrix of $\hat{Q}_X - \hat{Q}_Y$ is given by $\frac{1}{T}H\Lambda H'$. Various estimators of $\frac{1}{T}H\Lambda H'$ are given below.

[6]The MBB will be discussed in Section 3.4.

3.2 Test for First-Degree Stochastic Dominance

To test XD_1Y, *i.e.*, to test H_0: $Q_X - Q_Y \geq 0$ against H_a: $Q_X - Q_Y \ngeq 0$, we employ a version of the general Wald test for equality and inequality restrictions developed by Kodde and Palm (1986) and Wolak (1989a and 1989b). The following lemma provides a framework for both FSD and SSD tests, and is a special case of the Kodde and Palm result.

Lemma 2 *If a $k \times 1$ vector of parameters Q_Z can be consistently estimated by $\hat{Q}_Z$ based on a sample of size T such that $\sqrt{T}(\hat{Q}_Z - Q_Z) \xrightarrow{d} N(0, \Lambda)$, then for testing H_0: $h(Q_Z) \geq 0$ against H_a: $h(Q_Z) \ngeq 0$, the test-statistic D is defined as $D = \| \hat{\gamma} - \tilde{\gamma} \|_\Sigma = (\hat{\gamma} - \tilde{\gamma})'\Sigma^{-1}(\hat{\gamma} - \tilde{\gamma})$, where $\hat{\gamma} = \sqrt{T}h(\hat{Q}_Z)$ and $\tilde{\gamma} = \sqrt{T}h(\tilde{Q}_Z)$. $\hat{\gamma}$ is an unrestricted estimator and has large sample variance-covariance matrix $\Sigma = (\partial h/\partial Q_Z')\Lambda(\partial h'/\partial Q_Z)$. $\tilde{\gamma}$ is a restricted estimator solving $\min_\gamma(\hat{\gamma} - \gamma)'\Sigma^{-1}(\hat{\gamma} - \gamma)$ subject to the constraint $\gamma \geq 0$. D has a large sample distribution*

$$sup_{\gamma \geq 0} Pr(D \geq q|\Sigma) = \sum_{i=0}^{K} Pr[\chi^2(K - i) \geq q]W(K, i, \Sigma)$$

with W denoting the probability that i of the K elements of $\tilde{\gamma}$ are strictly positive, and q denoting the critical value.

Proof. See Appendix A. □

The upper and lower-bounds for the critical values for testing inequality restrictions are provided by Kodde and Palm (1986). The reason for computing the upper- and lower-bounds for the critical value is that computing the weights (W) can be nontrivial, involving the evaluation of K-multiple integrals for which closed forms are only available for a small K.[7] Kodde and Palm (1986) provide a partial solution to this problem by computing the upper- and lower-bound critical values that do not require computation of the weights. These bounds are given by

$$\alpha_l = \frac{1}{2} Pr(\chi_1^2 \geq q_l), \tag{12}$$

[7] Kudo (1963) shows the closed form expressions for the weights when $K \leq 4$. Shapiro (1985) develops alternative closed form formulae for the weights when $K = 4$. Bohrer and Chow (1978) give an algorithm of computing these weights up to the case where $K = 10$. However, it is recognized that the these numerical methods will be prohibitively expensive or intractable when $K \geq 8$.

and

$$\alpha_u = \frac{1}{2}Pr(\chi^2_{K-1} \geq q_u) + \frac{1}{2}Pr(\chi^2_K \geq q_u), \tag{13}$$

where q_l and q_u are the lower- and upper-bounds, respectively, for the critical values of the test-statistic. A lower-bound for the critical value is obtained by choosing a significance level α and setting degrees of freedom (df) equal to one. An upper-bound for the critical value is obtained by choosing a significance level α and setting df equal to K. Decision rules based on the statistic D are: if D exceeds the upper-bound value, reject H_0; and if D is smaller than the lower-bound value, do not reject H_0. If D is in the inconclusive region, Monte Carlo simulations, which are suitable to the case where $K \geq 8$, suggested by Wolak (1989b) should be used to compute the weights (see Appendix B).

Theorem 2 *Under Assumptions 1 and 2 , the variance-covariance matrix of* $\hat{Q}_X - \hat{Q}_Y$ *is* $\frac{1}{T}H\Lambda H'$, *where* Λ *is given by Lemma 1. Under* $H_0 : Q_X - Q_Y \geq 0$, *the test-statistic for FSD, denoted* c_1, *is given by:*

$$c_1 = \Delta'[\frac{1}{T}H\hat{\Lambda}H']^{-1}\Delta, \tag{14}$$

where $\Delta = [(\hat{Q}_X - \hat{Q}_Y) - (\tilde{Q}_X - \tilde{Q}_Y)]$; $\hat{Q}_X$, $\hat{Q}_Y$, *and* $\hat{\Lambda}$ *are the unrestricted estimates while* $\tilde{Q}_X$ *and* $\tilde{Q}_Y$ *are the restricted estimates minimizing*

$$[(\hat{Q}_X - \hat{Q}_Y) - (Q_X - Q_Y)]'[\frac{1}{T}H\hat{\Lambda}H']^{-1}[(\hat{Q}_X - \hat{Q}_Y) - (Q_X - Q_Y)] \tag{15}$$

$$s.t. \ \ (Q_X - Q_Y) \geq 0.$$

The test-statistic c_1 *is asymptotically distributed as a weighted sum of* χ^2 *random variables with different degrees of freedom i.e.*

$$\begin{aligned} & sup_{(Q_X - Q_Y)\geq 0} \quad Pr[c_1 \geq q | \frac{1}{T}H\Lambda H'] \\ = & \sum_{i=0}^{K} Pr[\chi^2(K-i) \geq q] W[K, i, \frac{1}{T}H\Lambda H']. \end{aligned} \tag{16}$$

The decision rules based on the statistic c_1 *are the same as those for the statistic* D *in Lemma 2.*

Proof. This result is a consequence of Lemmas 1 and 2. □

The test-statistic, c_1, employs the unrestricted estimates $\hat{Q}_X - \hat{Q}_Y$ and $\frac{1}{T}H\hat{\Lambda}H'$, and the restricted estimates $\tilde{Q}_X - \tilde{Q}_Y$ which are estimated by solving the restricted nonlinear optimization problem in equation (15).

3.3 Test for Second-Degree Stochastic Dominance

The SSD null hypothesis may be written as: H_0: $\Psi_X - \Psi_Y \geq 0$ against H_a: $\Psi_X - \Psi_Y \ngeq 0$.

Definition 4 *The cumulative quantile generator, B, is defined as a $(K \times K)$ lower triangular matrix with every non-zero element equal to unity, i.e.,*

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Given K, B premultiplies $\hat{Q}_j$ $(j = X, Y)$ yielding a K-variate vector of cumulative sample quantiles, i.e.,

$$\begin{aligned} B\hat{Q}_j &= [\sum_{i=1}^{p_1} \hat{Q}_j(p_i), \ldots, \sum_{i=1}^{p_K} \hat{Q}_j(p_i)]' \\ &= (\hat{\Psi}_j(p_1), \ldots, \hat{\Psi}_j(p_K))' \\ &= \hat{\Psi}_j, \end{aligned} \tag{17}$$

where $p_0 = 0$, $p_K = 1$ and $p_{s+1} - p_s = p_{l+1} - p_l$ for all $s, l = 0, 1, \ldots, K-1$.

Theorem 3 *Under Assumptions 1 and 2, the variance-covariance matrix of $B(\hat{Q}_X - \hat{Q}_Y)$ is $\frac{1}{T} BH\Lambda H'B'$, where Λ is given by Lemma 1. Under H_0: $B(Q_X - Q_Y) \gtrsim 0$, the test-statistic for SSD, denoted c_2, is given by:*

$$c_2 = (B\Delta)'[\frac{1}{T} BH\hat{\Lambda}H'B']^{-1}(B\Delta), \tag{18}$$

where $B\Delta = B[(\hat{Q}_X - \hat{Q}_Y) - (\tilde{Q}_X - \tilde{Q}_Y)]$. $\hat{Q}_X$, $\hat{Q}_Y$, and $\hat{\Lambda}$ are the unrestricted estimators while $\tilde{Q}_X$ and $\tilde{Q}_Y$ are the restricted estimators minimizing

$$\{B[(\hat{Q}_X - \hat{Q}_Y) - (Q_X - Q_Y)]\}'\{\frac{1}{T} BH\hat{\Lambda}H'B'\}^{-1}\{B[(\hat{Q}_X - \hat{Q}_Y) - (Q_X - Q_Y)]\} \tag{19}$$

$$s.t. \;\; B(Q_X - Q_Y) \gtrsim 0.$$

c_2 is asymptotically distributed as the weighted sum of χ^2 distributions i.e.:

$$\begin{aligned} sup_{[B(Q_X - Q_Y)] \geq 0} \quad & Pr[c_2 \geq q | \frac{1}{T} BH\Lambda H'B'] \\ = & \sum_{i=0}^{K} Pr[\chi^2(K-i) \geq q] W[K, i, \frac{1}{T} BH\Lambda H'B']. \end{aligned} \tag{20}$$

The decision rules based on the statistic c_2 are the same as those for the statistic D in Lemma 2.

Proof: This result is derived from Lemmas 1 and 2, Definition 4 acting as a linear transformation. □

As with the FSD test-statistic c_1, the SSD test-statistic c_2 employs the unrestricted estimates $B(\hat{Q}_X - \hat{Q}_Y)$ and $\frac{1}{T} BH\hat{\Lambda}H'B'$, and the restricted $B(\tilde{Q}_X - \tilde{Q}_Y)$, which are estimated by solving the restricted nonlinear optimization problem in equation (19). [8]

Several remarks should be made concerning the test-statistics c_1 and c_2. Although these test-statistics have desirable asymptotic properties, their performance in finite samples is practically important. Monte Carlo evidence presented in Xu (1994) indicates that for samples of size 400, these tests have power to distinguish economically significant deviations from the first- and second-degree dominance null hypotheses, although the tests do not have much power when the alternative under consideration is close to equality of distribution or quantile functions. In addition, if the random variables under consideration are positively correlated, the power of the tests improves substantially.

In empirical work, a choice of K must be made. If K is too small, the comparison will be made based on relatively large intervals in the support, and this may blur the true relation between the two quantile functions. On the other hand, it should be noted that since the sample quantiles from a given sample will be positively correlated, there will be some point at which increasing K will produce no gain in power. For the empirical work presented in this paper, K is chosen to be 20.

3.4 Moving-Block Bootstrap (MBB) Estimation

In order to construct FSD and SSD test-statistics, consistent estimates of $\frac{1}{T} H\Lambda H'$ are required. Given the complexity of Λ under the general conditions assumed here, we

[8] It should be noted that Δ in equation (14) and Δ in equation (18) are not identical because the two are computed from different restricted optimization procedures.

suggest the use of a computationally convenient resampling procedure, the moving-block bootstrap (MBB), to provide consistent estimates. To explain how the MBB works, it is useful to first consider the standard bootstrap which randomly resamples with replacement from a given sample of observations, as many times, N, as necessary. The variance of the statistic of interest is then estimated using the sample variance of the statistics calculated over the bootstrap replications. Rather than resampling individual observations, the MBB randomly resamples blocks of observations with replacement from the original data set. Let $\{Z_t\}_{t=1}^T$ be a finite sample of a sequence of stationary ϕ -mixing random vectors. Let $\hat{Q}_X - \hat{Q}_Y$ be the estimator of the true population parameter of interest, $Q_X - Q_Y$. Denote the moving blocks as $B_1, \cdots, B_{T-b+1}$, where b is the size of each block and B_j stands for the block consisting of b consecutive observations starting from Z_j, i.e., $B_j = \{Z_j, Z_{j+1}, \ldots, Z_{j+b-1}\}$. For each moving block bootstrap replication, $\hat{Q}_{X*} - \hat{Q}_{Y*}$ can be computed. If the resampling takes place N times, then $s = 1, 2, \ldots, N$ and the empirical sampling distribution of $\hat{Q}_X - \hat{Q}_Y$ can be constructed, along with various statistics associated with the distribution. In particular, the variance-covariance matrix of $\hat{Q}_X - \hat{Q}_Y$ can be computed using the sample variances and covarainces of $\hat{Q}_{X*} - \hat{Q}_{Y*}$.

Consistency for the MBB estimation is achieved if the number of observations in each block, b, approaches infinity with T in such a way that the number of moving blocks, $k = [T/b]$, also approaches infinity with T. In general, larger values for b are necessary to capture stronger dependence.

4 Two Special Cases

Two special cases of Theorems 2 and 3 are given in this section as examples of the general test-statistics when the data structure becomes more and more restrictive. In the first special case, Assumption 1 is replaced with:

Assumption 3 *The sample observations from the joint distribution, F_Z, are i.i.d..*

Assumption 3 represents $\{Z_t\}$ as an *i.i.d.* sequence of random variables that are jointly dependent. This type of assumption would be useful for measuring changes in income distributions, where the same panel of respondents is sampled at different points in time.

Under Assumptions 2 and 3, as $T \to \infty$,

$$\sqrt{T}(\hat{Q}_Z - Q_Z) \xrightarrow{d} N(0, \Lambda), \tag{21}$$

where the elements of Λ are given by

$$\frac{F_{js}(Q_j(p_i)), Q_s(p_t)) - p_i p_t}{f_j(Q_j(p_i)) f_s(Q_s(p_t))}$$

for $j, s = X, Y \quad (j \neq s)$, and $i, t = 1, 2, \ldots, K$;

$$\frac{p_i(1 - p_t)}{f_j(Q_j(p_i)) f_j(Q_j(p_t))}$$

for $j = X, Y \; (j = s)$, and $i, t = 1, 2, \ldots, K$ [see Siddiqui (1960) and Weiss (1964)]. In this case, $\hat{\Lambda}$ can be consistently estimated using standard nonparametric density estimation techniques, and the tests follow the steps outlined in the Sections 3.2 and 3.3.[9]

Another special case restricts the data structure even further. We may replace Assumption 1 with:

Assumption 4 *The sample observations for each random variable are i.i.d. and X and Y are independent.*

This assumption would be appropriate for independent income distribution samples. Under Assumptions 2 and 4, as $T \to \infty$,

$$\sqrt{T}(\hat{Q}_Z - Q_Z) \xrightarrow{d} N(0, \Lambda), \tag{22}$$

where $\Lambda = \begin{bmatrix} \Lambda_X & 0 \\ 0 & \Lambda_Y \end{bmatrix}$ with elements in Λ_j, $j = X, Y$ given by

$$\frac{p_i(1 - p_t)}{f_j(Q_j(p_i)) f_j(Q_j(p_t))}$$

for $j = X, Y$, and $i, t = 1, 2, \ldots, K$ [see Mosteller (1946)]. As a result, $\frac{1}{T} H \Lambda H' = \frac{1}{T}(\Lambda_X + \Lambda_Y)$, and tests are even simpler than in the previous case.

Assumptions 3 and 4 are common in the income distribution literature. Nevertheless, income distribution data are not usually obtained from simple random samples. It is possible to extend these results to the case of complex sampling designs used in finite population sampling. In this instance, inferences from the sample to the finite population concerning first- and second-degree stochastic dominance are based on sample quantiles and estimated variances which take into account the sampling design. The asymptotic distribution of the empirical distribution function and the sample quantiles from complex survey data has been developed by Fransisco and Fuller (1991), and test statistics that are analagous to those presented above follow immediately under their assumptions. Extensions to allow for dependence are also straightforward.

[9] Useful descriptions are provided in Silverman (1986), Ullah (1988), and Izenman (1992).

5 An Empirical Example

The existence of term premia in returns on the U.S. Treasury bills have long been recognized as an important feature of the term structure of the interest rates [Roll (1970,1971), Fama (1976)]. While the term premia increase with terms-to-maturity, the variance of real holding period returns also increases. In this context, stochastic dominance criteria are very useful for evaluating the economic significance of term premia, because these criteria do not impose any restrictions on either the types of utility functions or the forms of probability distributions.

A general conclusion from previous research (Levy and Brooks, 1989) is that first-degree stochastic dominance criterion does not provide discriminatory information concerning the relative rankings of the one- to sixth-month Treasury bills. However, this conclusion was not made on the basis of a statistical test that took sampling variability into account. In this example, the dominance relationships among the real holding period returns of U.S. Treasury bills will be evaluated based on the proposed test procedures.

We consider the yields of one- and two-month Treasury bills for the period of 1952:02–1987:02.[10] The yields are transformed to nominal holding period returns based on Shiller (1990). These are then transformed into the real holding period returns by substracting the rate of price inflation using the consumer price index. These real holding period returns are denoted as $h^r(m)$, $m = 1, 2$. where m represents the number of month(s) to maturity. A more complete analysis of the entire term structure is provided by Fisher *et al.* (1994).

Tables 1 and 2 present some basic statistical properties of the real return time series. In particular, Table 1 provides results from the augmented Dickey-Fuller tests showing that real returns do not have a unit root. Table 2 indicates that both the mean and variance of the real returns increase as the term-to-maturity becomes longer. In addition, skewness and kurtosis also increase with the term-to-maturity. Inspection of the sample autocorrelation functions for each real return (see Table 3) indicates that real returns have a non-trivial serial correlation structure, and that the autocorrelations die out after lag 30. As a precautionary measure, we tested each time series for long-range dependence using the modified rescaled range test developed by Lo (1991). In general, we found no evidence of long-range dependence.[11] Finally, the contemporaneous correlation between the real returns is quite high (0.96835), and accounting for this positive dependence will be important for the test results. We report test results for MBB block sizes of $b = 30$, 40, 50, and 60, using 200 replications and 20 equally spaced quantiles.

[10]The data are from J. H. McCulloch and are reprinted in the Appendix of Shiller (1990).

[11]See Fisher *et al.* (1994) for details.

Tables 4 and 5 show the test results of FSD and SSD relations among the selected pairs of the real returns. For testing the null hypotheses of FSD and SSD at a significance level $\alpha = 0.05$, each null hypothesis will be rejected if the test-statistic is greater than 30.841; it will not be rejected if the test-statistic is less than 2.706. If the test-statistic falls into inconclusive region, then the weights used for computing the test-statistic must be computed to get the p-value.

Figure 5 plots the quantile functions for the two time series. While the Figure suggests that the two-month real return dominates the one-month real return, a test is required to determine whether the distance is statistically significant. We evaluate both $h^r(1)D_i h^r(2)$ (i=1,2) and $h^r(2)D_i h^r(1)$ ($i = 1,2$). The test-statistics in Table 4 show that the null hypothesis of $h^r(2)D_1 h^r(1)$ cannot be rejected at the 5% significance level, while the test statistic in the opposite direction falls in the inconclusive region. Similar remarks can be made for the test results for $h^r(2)D_2 h^r(1)$ and $h^r(1)D_2 h^r(2)$ in Table 5.[12]

For test statistics in the inconclusive region, we perform Monte Carlo simulations as suggested by Wolak (1989b) for b=30 and b=40.[13] For the FSD test-statistics c_1, it is found that the p-value of the test-statistic (12.564) at $b = 30$ is 0.023. The p-value of the test-statistic (11.069) at $b = 40$ is 0.041. For the SSD test-statistics c_2, it is found that the p-value of the test-statistic (10.087) at $b = 30$ is 0.055. The p-value of the test-statistic (8.719) at $b = 40$ is 0.098. Clearly, these p-values imply that the null hypothesis of $h^r(1)D_1 h^r(2)$ should be rejected at the 5% or 10% significant levels. The direct interpretation of the above test results is: at the given significance level, (i) the hypothesis that the two-month real return dominates, in both first and second-degree, the two-month real return cannot be rejected; and (ii) the hypothesis that the one-month real return dominates, in both first and second-degree, the two-month real return should be rejected. It appears that the two-month real return is not only at least as preferred as, but also strictly preferred to the one-month real return. This result differs from that of Levy and Brooks (1989).

[12] The stochastic dominance relationship between the two assets could be one of the four possibilities: (i) X strongly dominates Y; (ii) Y strongly dominates X; (iii) X and Y are identical; and (iv) neither X dominates Y nor Y dominates X. The test procedures proposed here will test for weak form of stochastic dominance. This will reduce the four possibilities into three: (a) X weakly dominates Y; (b) Y weakly dominates X; and (c) neither X dominates Y nor Y dominates X. Logically, if both (a) and (b) hold, then (iii) is true; if (a) holds but (b) does not, then (i) is true.

[13] For this computation, $T = 400$ and $N = 1000$.

6 Concluding Remarks

It has been noted that existing test procedures for stochastic dominance suffer from several weaknesses. This paper proposes new distribution-free test procedures that properly specify the dominance relationship under the null hypothesis, and that are valid under very general conditions concerning the data. As an example, we evaluated the dominance relationships between one- and two-month real U.S. T-bill yields, and found that the two-month yield dominated the one-month yield in both the first and the second degree.

Regarding future research, it is important to note that convenient applications of these procedures with financial data will typically require use of the MBB, and that further research on the performance of the MBB in empirically relevant situations is warranted. It would also be useful to compare the results of these test procedures with the results from alternative tests, within the context of measuring changes in income distributions.

Appendix A Derivation of Lemma 2

As shown in Kodde and Palm, a vector of parameters of interest Q_Z is formulated in terms of K independent continuous functions $h(Q_Z)$, which are differentiable in some open neighborhood of the true parameters Q_Z. The hypothesis to be tested can be written as H_0: $h_1(Q_Z) = 0$, and $h_2(Q_Z) \geq 0$ against H_a: $h_1(Q_Z) \neq 0$, and $h_2(Q_Z) \ngeq 0$. The dimensions of the partition of $h(Q_Z)$ into $h_1(Q_Z)$ and $h_2(Q_Z)$ are r and $K - r$ respectively.

It is assumed that Q_Z can be consistently estimated by $\hat{Q}_Z$ such that $\sqrt{T}(\hat{Q}_Z - Q_Z) \xrightarrow{d} N(0, \Lambda)$. Thus, by the delta method, $\sqrt{T}(h(\hat{Q}_Z) - h(Q_Z)) \xrightarrow{d} N(0, \Sigma)$, where $\Sigma = (\partial h/\partial Q_Z')\Lambda(\partial h'/\partial Q_Z)$.

The functions of parameters, $h(Q_Z)$, can be transformed into new parameter vectors $\gamma = (\gamma_1', \gamma_2')'$ and $\hat{\gamma} = (\hat{\gamma}_1', \hat{\gamma}_2')'$, where $\gamma_i = \sqrt{T}h_i(Q_Z)$ and $\hat{\gamma}_i = \sqrt{T}h_i(\hat{Q}_Z)$.

It is shown that for H_0: $\gamma_1 = 0$, and $\gamma_2 \geq 0$ against H_a : $\gamma_1 \neq 0$, and $\gamma_2 \ngeq 0$, the Wald test-statistic is

$$\begin{aligned} D &= \| \hat{\gamma} - \tilde{\gamma} \|_\Sigma \\ &= \hat{\gamma}_1'\Sigma_{11}^{-1}\hat{\gamma}_1 \\ &\quad +(\hat{\gamma}_2 - \tilde{\gamma}_2 - \Sigma_{21}\Sigma_{11}^{-1}\hat{\gamma}_1)'(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}(\hat{\gamma}_2 - \tilde{\gamma}_2 - \Sigma_{21}\Sigma_{11}^{-1}\hat{\gamma}_1). \end{aligned}$$

where $\hat{\gamma}$ is an unrestricted consistent estimator, $\tilde{\gamma}_1 = 0$ and $\tilde{\gamma}_2$ is the solution of

$$\min_{\gamma_2 \geq 0}(\hat{\gamma}_2 - \gamma_2 - \Sigma_{21}\Sigma_{11}^{-1}\hat{\gamma}_1)'(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}(\hat{\gamma}_2 - \gamma_2 - \Sigma_{21}\Sigma_{11}^{-1}\hat{\gamma}_1).$$

For the maximum under the null hypothesis the large sample distribution is

$$sup_{\gamma_2 \geq 0} Pr(D \geq q|\Sigma) = \sum_{i=0}^{K-r} Pr[\chi^2(K - i) \geq q]W(K - r, i, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}),$$

with W denoting the probability that i of the $K - r$ elements of $\tilde{\gamma}_2$ are strictly positive, and q denoting the critical value.

Now consider the hypotheses H_0: $\gamma_2 \geq 0$ and H_a: $\gamma_2 \ngeq 0$. This sets $r = 0$. The dimension of γ_2 is K instead of $K - r$. In this case, γ_1 is not in the parameter space, and Σ_{12}, Σ_{11} and Σ_{21} become irrelevant. Simply rename γ_2 as γ, $\hat{\gamma}_2$ as $\hat{\gamma}$, $\tilde{\gamma}_2$ as $\tilde{\gamma}$, and Σ_{22} as Σ. Then we have the following result.

For H_0: $h(Q_Z) \geq 0$ against H_a: $h(Q_Z) \ngeq 0$, the test-statistic D is defined as $D = \| \hat{\gamma} - \tilde{\gamma} \|_\Sigma = (\hat{\gamma} - \tilde{\gamma})'\Sigma^{-1}(\hat{\gamma} - \tilde{\gamma})$, where $\hat{\gamma} = \sqrt{T}h(\hat{Q}_Z)$ and $\tilde{\gamma} = \sqrt{T}h(\tilde{Q}_Z)$. $\tilde{\gamma}$

is an unrestricted estimator and has the large sample variance-covariance matrix $\Sigma = (\partial h/\partial Q'_{\bar{z}})\Lambda(\partial h'/\partial Q_{\bar{z}})$. $\tilde{\gamma}$ is a restricted estimator solving $\min_\gamma(\hat{\gamma}-\gamma)'\Sigma^{-1}(\hat{\gamma}-\gamma)$ subject to the constraint $\gamma \geq 0$. D has a large sample distribution

$$sup_{\gamma\geq 0} Pr(D \geq q|\Sigma) = \sum_{i=0}^{K} Pr[\chi^2(K-i) \geq q]W(K,i,\Sigma)$$

with W denoting the probability that i of the K elements of $\tilde{\gamma}$ are strictly positive, and q denoting the critical value. When $i = K$, $Pr[\chi^2(0) \geq q] = 0$, for $q > 0$.

Wolak (1989a and 1989b) provides the same result in a different context.

Appendix B Computation Weights Using Monte Carlo Simulation

If D is in the inconclusive region, Monte Carlo simulations, which are suitable to the case where $K \geq 8$, suggested by Wolak (1989b) should be used to compute the weights. The following steps should be taken: (i) draw N random samples of size T from a multivariate normal distribution with mean zero and the covariance matrix $\hat{\Sigma}$, that is a consistent estimate of Σ; (ii) for each sample, compute $\tilde{\gamma}$ using the nonlinear programming; (iii) evaluate the number of positive elements for each $\tilde{\gamma}$; and (iv) compute the relative frequency of the i ($i = 0, \ldots, K$) positive elements in N iterations. When these weights are computed, the probability of the statistic D greater than or equal to q is the weighted probabilities of χ^2 random variables having the value greater than or equal to q. Then D can be compared with the critical value corresponding to the chosen significance level α.

References

Anderson, G., 1994, Nonparametric tests of stochastic dominance in income distribution, Mimeo, Department of Economics, University of Toronto, Toronto, Canada.

Baillie, R. and T. Bollerslev, 1990, A multivariate generalized ARCH approach to modelling risk premia in forward foreign exchange rate markets, Journal of International Money and Finance, 9, 309–324.

Bawa, V. S., 1975, Optimal rules for ordering uncertain prospects, Journal of Financial Economics 2, 95–121.

Beach, C. M. and R. Davidson, 1983, Distribution-free statistical inference with Lorenz curves and income shares, Review of Economic Studies, 723–735.

Bishop, J. A., J. P. Formby, and P. D. Thistle, 1989, Statistical inference, income distributions, and social welfare, Research on Economic Inequality 1, 49–82.

Bishop, J. A., S. Chakraborti, and P. D. Thistle, 1989, Asymptotically distribution-free statistical inference for generalized Lorenz curves, Review of Economics and Statistics 71, 725–727.

Bohrer, R. and W. Chow, 1978, Weights of one-sided multivariate inference, Applied Statistics 27, 100–104.

Deshpande, J. V. and H. Singh, 1985, Testing for second order stochastic dominance, Communications in Statistics: Theory and Methods 14, 887–897.

Efron, B., 1982, The jackknife, the bootstrap, and other resampling plans, SIAM.

Efron, B. and R. Tibshirani, 1986, Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy, Statistical Science 1, 54–77.

Engle, R. F., 1982, Autoregressive conditional heteroscadasticity with estimates of the variance of United Kingdom inflation, Econometrica 50, 987–1007.

Fama, E., 1976, Inflation uncertainty and expected returns on Treasury bills, Journal of Political Economy 84, 427–488.

Fisher, G., D. Willson, and K. Xu, 1994, Are term premia economically significant? Mimeo, Department of Economics, Dalhousie University, Halifax, Canada.

Francisco, C. and W. A. Fuller, 1991, Quantile estimation with a complex survey design, Annals of Statistics 19, 454–469.

Hadar, J. and W. R. Russell, 1969, Rules for ordering uncertain prospects, American Economic Review 59, 25–34.

Hanoch, G. and H. Levy, 1969, The efficiency analysis of choices involving risk, Review of Economic Studies 36, 335–346.

Izenman, A. J., 1991, Recent developments in nonparametric density estimation, Journal of the American Statistical Association 86, 205–224.

Kodde, D. A. and F. C. Palm, 1986, Wald criteria for jointly testing equality and inequality restrictions, Econometrica 50, 1243–1248.

Kroll, D. and H. Levy, 1980, Stochastic dominance criteria: A review and some new evidence, Research in Finance 2, JAI Press, 163–227.

Kudo, A., 1963, A multivariate analogue of the one-sided-test, Biometrika 50, 403–418.

Kunsch, H., 1989, The jackknife and the bootstrap for general stationary observations, Annals of Statistics 17, 1217–1241.

Levy, H., 1992, Stochastic dominance and expected utility, Management Science 38, 555–593.

Levy, H. and R. Brooks, 1989, An empirical analysis of term premiums using stochastic dominance, Journal of Banking and Finance 13, 245–260.

Levy, H. and G. Hanoch, 1970, Relative effectiveness of efficiency criteria for portfolio selection, Journal of Financial and Quantitative Analysis, 5, 63–76.

Levy, H. and Y. Kroll, 1978, Ordering uncertain options with borrowing and lending, Journal of Finance 33, 553–573.

Levy, H. and Y. Kroll, 1979, Efficiency analysis with borrowing and lending: Criteria and Their Effectiveness, Review of Economics and Statistics 61, 25–31.

Liu, R. Y. and K. Singh, 1992, Moving blocks jackknife and bootstrap capture weak dependence, in R. LePage, and L. Billard (ed.), Exploring the Limits of Bootstrap, John Wiley and Sons, 225–248.

Lo, Andrew W., 1991, Long-term memory in stock market prices, Econometrica 59, 1279–1313.

Lo, A. W. and A. C. MacKinlay, 1988, Stock market prices do not follow random walks: Evidence from a simple specification test, Review of Financial Studies 1, 41–46.

McFadden, D., 1989, Testing for stochastic dominance, in T. B. Fomby and T. K. Seo (ed.), Studies in the Economics of Uncertainty: In Honor of Josef Hadar, Springer Verlag, 113–134.

Mosteller, F., 1946, On some useful 'inefficient' statistics, Annals of Mathematical Statistics 17, 377–408.

Parzen, E., 1962, On estimation of a probability density function and mode, Annals of Mathematical Statistics 33, 1065–1076.

Porter, R. B., J. R. Wart and D. L. Ferguson, 1973, Efficient algorithms for conducting stochastic dominance tests on large numbers of portfolios, Journal of Financial and Quantitative Analysis 8, 71–82.

Rao, C. R., 1965, Linear Statistical Inference and Its Applications, Wiley.

Roll, R., 1970, The Behavior of Interest Rates: The Application of the Efficient Market Model to U.S. Treasury Bills, Basic Books.

Roll, R., 1971, Investment diversification and bond maturity, Journal of Finance 26, 51–66.

Rothschild, M. and J. E. Stiglitz, 1970, Increasing risk: I. A definition, Journal of Economic Theory 2, 225–243.

Sen, P. K., 1972, On the Bahadur representation of sample quantiles for sequences of ϕ-mixing random variables, Journal of Multivariate Analysis 2, 77–95.

Serfling, R. J., 1980, Approximation theorems of mathematical statistics, Wiley.

Shapiro, A., 1985, Asymptotic distribution of test statistics in the analysis of moment structures under inequality constraints, Biometrika 72, 133–144.

Shiller, R., 1990, The term structure of interest rates, in B. M. Friedman and F. H. Hahn (ed.), Handbook of Monetary Economics, Vol. I, Chapter 13, 627–722.

Shorrocks, A. F., 1983, Ranking income distributions, Economica 50, 3–17.

Siddiqui, M. M., 1964, Distribution of quantiles in samples from a bivariate population, Journal of Research of the National bureau of Standards, B. Mathematics and Mathematical Physics, Vol. 64B, No.3, 145-150.

Silverman, B. W., 1986, Density Estimation for Statistics and Data Analysis, Chapman and Hall.

Stein, W. E. and R. C. Pfaffenberger, 1986, Estimation of error probabilities in stochastic dominance, Journal of Statistical Computation and Simulation 25, 137–165.

Tolley, H. D. and R. D. Pope, 1988, Testing for stochastic dominance, American Journal of Agricultural Economics 70, 694–700.

Ullah, A., 1988, Nonparametric estimation of econometric functionals, Canadian Journal of Economics 21, 625–658.

Vickson, R. G., 1977, Stochastic dominance tests for decreasing absolute risk-aversion: II. General-random variables, Management Science 23, 478–489.

Vickson, R. G. and Altman, 1977, On the relative effectiveness of stochastic dominance rules: Extension to decreasingly risk-averse utility functions, Journal of Financial and Quantitative Analysis 12, 73–84.

Weiss, L., 1964, On the asymptotic joint normality of quantiles from a multivariate distribution, Journal of Research of the National Bureau of Standards, B. Mathematics and Mathematical Physics, Vol. 68B, No.2, 65–66.

Wolak, F. A., 1989a, Local and global testing of linear and nonlinear inequality constraints in nonlinear econometric models, Econometric Theory 5, 1–35.

Wolak, F. A., 1989b, Testing inequality constraints in linear econometric models, Journal of Econometrics 41, 205–235.

Whitmore, G. A., 1970, Third degree stochastic dominance, American Economic Review 60, 457–459.

Whitmore, G. A. and M. C. Findlay (ed.), 1978, Stochastic Dominance: An Approach to Decision-Making Under Risk, Lexington Books.

Wilks, S. S., 1962, Mathematical Statistics, Wiley.

Xu, K. 1994, Dominance Testing in Economics and Finance, Unpublished PhD Thesis, Concordia University, Montreal, Canada.

Table 1: The Augmented Dickey-Fuller Tests on Real Returns: 1954:02–1987:01

Series	*Test Statistic t*	*Asymptotic Critical Value at 10%*	*l*
$h^r(1)$	-2.74	-2.57	8
$h^r(2)$	-3.15	-2.57	7

Note: $h^r(i)$, $i = 1, 2$ represent the real holding period returns of the i-month U.S. Treasury bills. The unit root tests indicate that all the test-statistics are less than the asymptotic critical value -2.57. $h^r(i)$, $i = 1, 2$, are stationary.

Table 2: Summary Statistics For Real Returns: 1954:02–1987:01

Series	*Mean*	*Variance*	*Skewness*	*Kurtosis*
$h^r(1)$	0.06780	0.08100	0.06921	1.90701
$h^r(2)$	0.11238	0.08584	0.14013	1.78988

Note: $h^r(i)$, $i = 1, 2$ represent the real holding period returns of the i-month U.S. Treasury bills.

Table 3: Autocorrelation Functions for Real Returns: 1954:02–1987:01

	lag 5	*lag* 10	*lag* 15	*lag* 20	*lag* 25	*lag* 30	*lag* 35	*lag* 40	*lag* 45	*lag* 50
$h^r(1)$	.313	.353	.281	.236	.089	.116	.042	.058	-.073	-.051
$h^r(2)$	.323	.338	.314	.241	.124	.022	.076	.061	-.062	-.041

Note: $h^r(i)$, $i = 1, 2$, represent the real holding period returns of the i-month U.S. Treasury bills.

Table 4: Test Statistics of First-Degree Stochastic Dominance c_1 with Different MBB Block Size b

H_0	$b = 30$	$b = 40$	$b = 50$	$b = 60$
$h^r(2)D_1h^r(1)$	0.000	0.000	0.000	0.000
$h^r(1)D_1h^r(2)$	12.564	11.069	10.726	10.930

Note: $h^r(i)D_1h^r(j)$ refers to the i-month real holding period return dominates, in the first-degree, the j-month real holding period return. At $\alpha = 0.05$, H_0, under which either $Q_Y(P) - Q_X(P) \gtrsim 0 \ \forall P \in [0,1]$ or $Q_X(P) - Q_Y(P) \gtrsim 0 \ \forall P \in [0,1]$, will be rejected if the test statistic is greater than 30.841; it will not be rejected if the test statistic is less than 2.706.

Table 5: Test Statistics of Second-Degree Stochastic Dominance c_2 with Different MBB Block Size b

H_0	$b = 30$	$b = 40$	$b = 50$	$b = 60$
$h^r(2)D_2h^r(1)$	0.000	0.000	0.000	0.000
$h^r(1)D_2h^r(2)$	10.087	8.719	6.360	7.707

Note: $h^r(i)D_2h^r(j)$ refers to the i-month real holding period return dominates, in the second-degree, the j-month real holding period return. At $\alpha = 0.05$, H_0, under which either $\Phi_Y(P) - \Phi_X(P) \gtrsim 0 \ \forall P \in [0,1]$ or $\Phi_X(P) - \Phi_Y(P) \gtrsim 0 \ \forall P \in [0,1]$, will be rejected if the test statistic is greater than 30.841; it will not be rejected if the test statistic is less than 2.706.

Figure 1: The stochastic dominance: XD_1Y

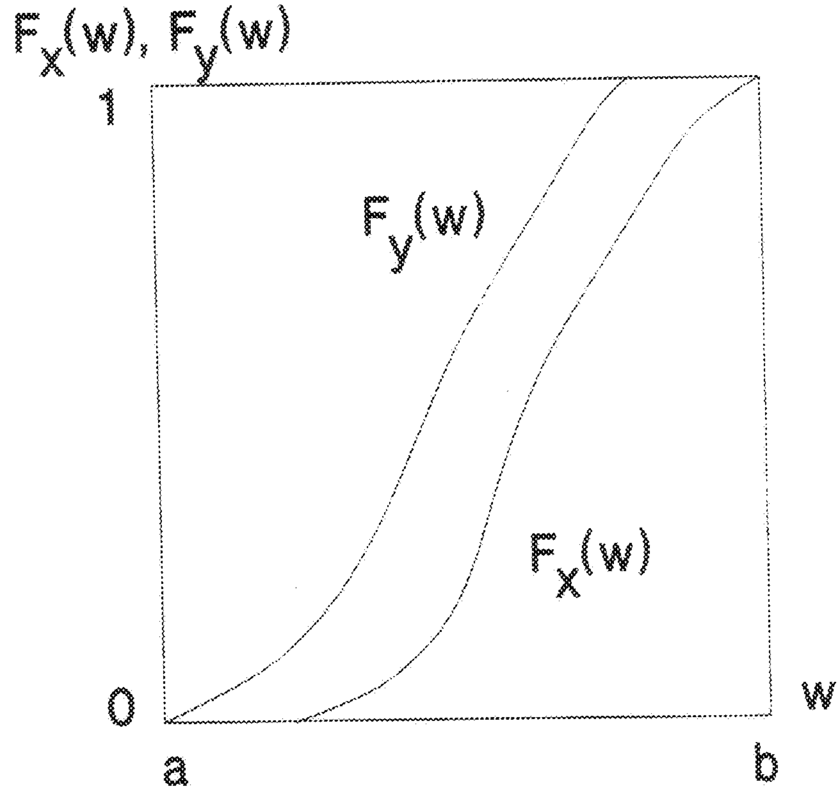

Figure 2: The stochastic dominance: XD_2Y

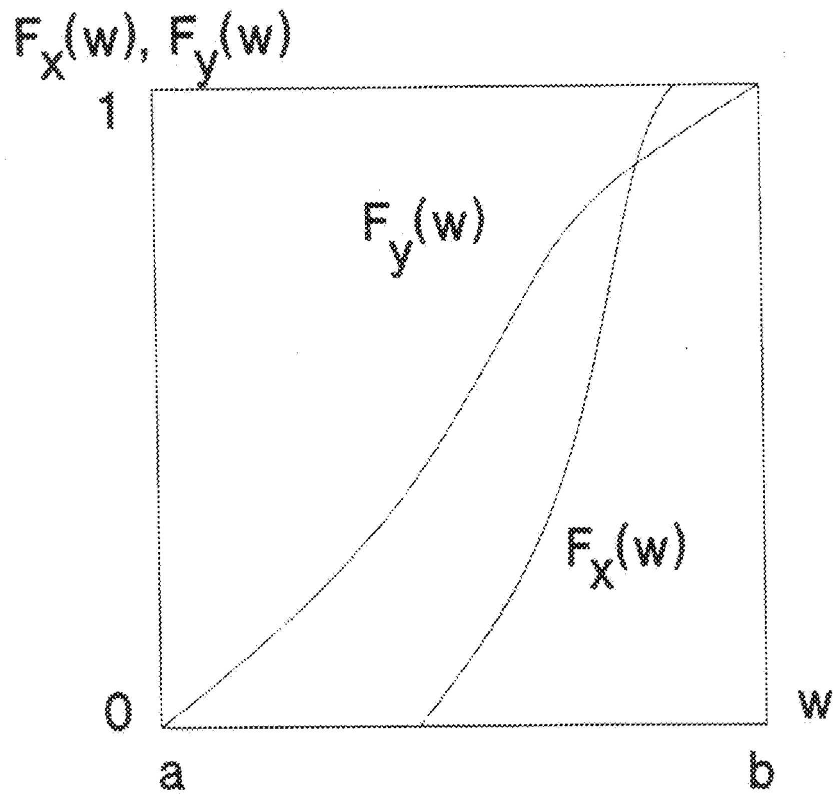

Figure 3: The quantile condition of stochastic dominance: XD_1Y

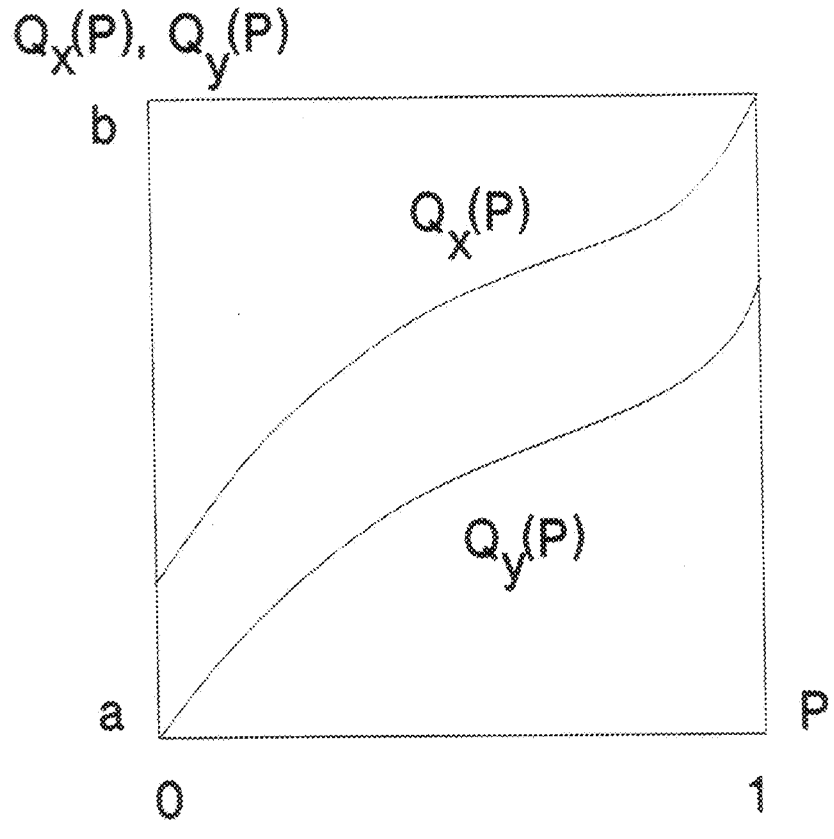

Figure 4: The quantile condition of stochastic dominance: XD_2Y

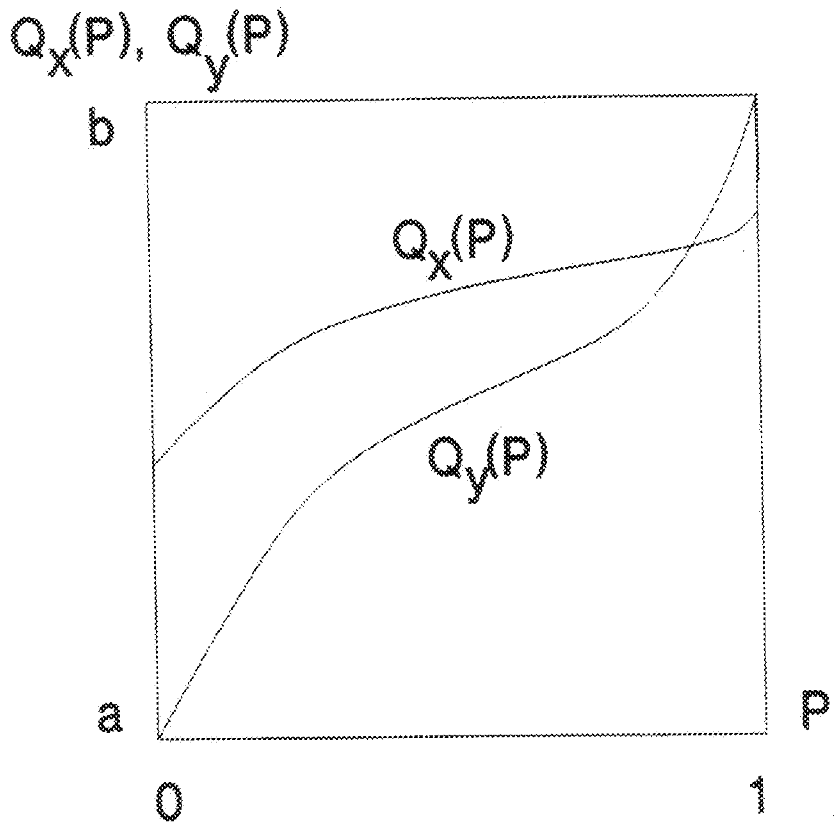

Figure 5: Empirical quantile functions of the one- and two-month real returns