The social welfare implications, decomposability, and geometry of the Sen family of poverty indices

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Abstract. In this paper, we propose a unified framework for the Sen indices of poverty intensity that shows an explicit connection between the indices and the common underlying social evaluation function. We also identify the common multiplicative decomposition of the indices that allows simple and similar geometric interpretations and easy numerical computation. JEL Classification: C000, H000, O150

Implications en termes de bien-être collectif, décomposabilité et géométrie de la famille d'indices de pauvreté à la Sen. Dans ce mémoire, les auteurs proposent un cadre d'analyse intégré des indices d'intensité de pauvreté à la Sen qui souligne la relation explicite entre ces indices et la fonction sociale d'évaluation commune qui les soustend. Les auteurs identifient aussi la décomposition multiplicative commune de ces indices qui permet d'en donner des interprétations géométriques simples et idoines et de faire des calculs numériques faciles.

1. Introduction

Since Sen proposed an axiomatic approach to poverty research and an index of poverty intensity in 1976, poverty measurement has become an active research agenda. A vast theoretical literature has developed over the years,[1] and applications

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1 See Zheng (1997) and Foster and Sen in Sen (1997), and the references therein, for recent comprehensive surveys.

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of the Sen index (or S index[2]) and the modified Sen index (or SST index[3]) have appeared in empirical poverty studies by Bishop, Formby, and Zheng (1997), Myles and Picot (2000), Osberg (2000), Osberg and Xu (1997, 1999, 2000), Rongve (1997), and Xu (1998), among others.

The Sen indices are based on a set of well-justified and commonly agreed axioms. However, although from a policy point of view it is also desirable to understand the meaning of the Sen indices in terms of social welfare evaluation, the social evaluation function that the Sen indices jointly share has not yet been explicitly summarized in the literature.[4] Bourguignon and Fields (1997) pointed out that poverty measures can be interpreted as gauging the social welfare losses when persons have low incomes. Blackorby and Donaldson (1978, 1980) and Chakravarty (1983, 1997) laid a solid ground for interpreting the social welfare meaning of many inequality and poverty indices. In this paper we will use their work to examine the Sen indices' underlying social welfare function.

From a policy point of view, it is also desirable to understand the relationship between the Sen indices and their contributing components (see, for example, and Birdsall and Londono 1997, Phipps 1999, among others). Poverty indices that exhibit *additive* decomposability, such as the indices proposed by Foster, Greer, and Thorbecke (1984), are often selected in empirical studies.[5] While the Sen indices and their generalizations such as BD index and C index (see Blackorby and Donaldson 1980 and Chakravarty 1983, respectively) do not satisfy this axiom in general,[6] they do have the property of *multiplicative* decomposability, as first briefly mentioned for the S index by Clark, Hemming, and Ulph (1981) for the S index and as examined for the SST index by Osberg and Xu (1997, 1999, 2000). In addition, as Bourguignon and Fields (1997) noted, some additive poverty measures may bias anti-poverty policy to paying attention to the richest of the poor. They noted that an appropriate poverty measure should facilitate a comprehensive evaluation of anti-poverty policy actions. Since the Sen indices have desirable ethical properties and are multiplicatively decomposable, the Sen indices and their decomposed components can be readily used to measure the multidimensional impacts of anti-poverty policy actions.

In the literature on income inequality, the Gini index is perhaps the most-used index of inequality, partly because it has a useful and intuitive geometric interpretation. The SST index has a useful and intuitive geometric interpretation (see Shor-

2 See Sen (1976). The index is called the S index in Sen (1997).

3 The index is called the modified Sen index in Shorrocks (1995) and Sen (1997). Shorrocks (1995) proposed the index. Zheng (1997) noted that the modified Sen index is identical to the limit of Thon's modified Sen index (Thon 1979, 1983). Thus, we also call it the Sen-Shorrocks-Thon index (see Osberg and Xu 1997, 1999, 2000).

4 Dalton (1920), in his pioneering paper, suggested that any measure of income inequality has an underlying social welfare function. This has been made precise by Kolm (1969), Atkinson (1970), and Sen (1973).

5 See chapter 7 of Chakravarty (1990) for a detailed survey of the additive decomposition of the poverty intensity indices.

6 The Chakravarty index with the symmetric means of order r ($r < 1$) is an exception.

rocks 1995; Jenkins and Lambert 1997; Osberg and Xu 1997, 2000; and Xu and Osberg 1998). The geometric interpretation of the S index, however, is less intuitive (see Sen 1976, 226). The common multiplicative decomposability of the Sen indices suggests that the decompositions must have similar useful and intuitive geometric interpretations.

In this paper, we examine the common underlying social evaluation function, multiplicative decomposability, and geometric interpretations of the Sen indices. In section 2, the notation and some basic concepts are introduced. In section 3, we discuss the common underlying social evaluation function, the common multiplicative decomposability, and useful and intuitive geometric interpretations of the Sen indices. Section 4 concludes.

2. Notation and some basic concepts

Let $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathsf{T}}$ be the income vector of a population of size n with (individual or family) incomes sorted in non-decreasing order, where 'T' is used to denote transposition of a matrix or a vector. Let $\tilde{\mathbf{y}}$ be $\mathbf{y}$ with incomes sorted in non-increasing order where the notation '$\sim$' (tilde) is used for sorting a vector $\mathbf{x}$ in opposite order. Let the poverty line be $z > 0$. Let the number of the poor be q. Hence the poverty rate H is q/n. A censored income vector is obtained by setting $\overset{*}{y}_i = y_i$ if $y_i < z$ and $\overset{*}{y}_i = z$ otherwise,[7] that is $\overset{*}{\mathbf{y}} = [\overset{*}{y}_1, \overset{*}{y}_2, \ldots, \overset{*}{y}_n]^{\mathsf{T}}$. The income vector of the poor, $\mathbf{y}_p = [y_1, y_2, \ldots, y_q]^{\mathsf{T}}$, is a truncated income vector generated from $\overset{*}{\mathbf{y}}$ by deleting zs. The average of an income vector $\mathbf{y}$ is given by $\bar{\mathbf{y}} = (1/n)\sum_{i=1}^{n} y_i$.

Define $\mathbf{1}$ as a column vector of ones with an appropriate dimension. The poverty gap ratio vector of the population is defined as $\mathbf{x} = (z - \overset{*}{\mathbf{y}})/z$, where the poor have poverty gap ratios $x_i = (z - y_i)/z$, $i = 1,2,\ldots,q$, and the non-poor have zero poverty gap ratios. Similarly, the poverty gap ratio vector of the poor is given by $\mathbf{x}_p = (z\mathbf{1} - \mathbf{y}_p)/z$, where the poor have poverty gap ratios $x_i = (z - y_i)/z$, $i = 1,2,\ldots,q$, and the non-poor's zero poverty gap ratios are excluded. Please note that the elements in both $\mathbf{x}$ and $\mathbf{x}_p$ are in non-increasing order. The average poverty gap ratio of the population (the poor) is denoted by $\bar{\mathbf{x}}$ ($\bar{\mathbf{x}}_p$) and $\bar{\mathbf{x}} = H\bar{\mathbf{x}}_p$.

To analyse the social welfare implication of the Sen indices, we need to utilize the concept of the equally-distributed-equivalent-income (EDEI), or the representative income proposed by Atkinson (1970), Kolm (1969), and Sen (1973). For a particular social evaluation function (SEF), an EDEI given to every individual could be viewed as identical in terms of social welfare to an actual income distribution. Let $W(\mathbf{y}) = \phi(\overline{W}(\mathbf{y}))$ be a homothetic (ordinal) SEF of income, with ϕ being an increasing function and $\overline{W}$ being a linearly homogeneous function. Let ξ be the EDEI and $\mathbf{1}$ be a column vector of ones with an appropriate dimension. Then,

7 We use the weak definition of the poor here – a poor person's income is less than the poverty line – as it is generally treated in the literature.

$W(\xi \cdot \mathbf{1}) = W(\mathbf{y})$ or $\overline{W}(\xi \cdot \mathbf{1}) = \overline{W}(\mathbf{y})$. Given that $\overline{W}$ is positively linearly homogeneous, EDEI is computed by $\xi = \overline{W}(\mathbf{y})/\overline{W}(\mathbf{1}) = \Xi(\mathbf{y})$. The SEF ($W$) and the EDEI ($\Xi$) have a one-to-one corresponding relationship.

For example, the Gini SEF is $\overline{W}_G(\mathbf{y}) = (1/n^2)\sum_{i=1}^{n}(2n-2i+1)y_i$.[8] Its corresponding EDEI function is

$$\Xi_G(\mathbf{y}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n-2i+1)y_i \tag{1}$$

or

$$\Xi_{\tilde{G}}(\tilde{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2i-1)\tilde{y}_i, \tag{2}$$

with $\Xi_G(\mathbf{y}) = \Xi_{\tilde{G}}(\tilde{\mathbf{y}})$.[9] The Gini SEF attaches a higher weight to a lower level of income and vice versa. The weight is determined by the rank of an income rather than the size of the income.[10]

The Gini index can be defined in terms of the Gini EDEI and the mean income as

$$G(\mathbf{y}) = 1 - \frac{\Xi_G(\mathbf{y})}{\bar{\mathbf{y}}} = 1 - \frac{1}{n^2\bar{\mathbf{y}}}\sum_{i=1}^{n}(2n-2i+1)y_i \tag{3}$$

or

$$\tilde{G}(\tilde{\mathbf{y}}) = 1 - \frac{\Xi_{\tilde{G}}(\tilde{\mathbf{y}})}{\bar{\mathbf{y}}} = 1 - \frac{1}{n^2\bar{\mathbf{y}}}\sum_{i=1}^{n}(2i-1)\tilde{y}_i, \tag{4}$$

where $\mathbf{y}$ ($\tilde{\mathbf{y}}$) has elements in non-decreasing (non-increasing) order.[11] Note that $G(\mathbf{y}) = \tilde{G}(\tilde{\mathbf{y}})$ in equations (3) and (4) are identical, but $G(\cdot)$ and $\tilde{G}(\cdot)$ have different functional forms and the elements in $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are sorted differently. Also note that

$$G(\mathbf{y}) = -G(\tilde{\mathbf{y}}) \tag{5}$$

(see Fei, Ranis, and Kuo 1978; and Xu and Osberg 2001a).

3. Common SEF and multiplicative decomposition

3.1. Common Gini SEF

The link between the S and SST indices can be better understood based on the BD and C indices introduced by Blackorby and Donaldson (1980) and Chakravarty

8 This is because $\overline{W}_G(\mathbf{1}) = (1/n^2)\sum_{i=1}^{n}(2n-2i+1) = 1$.
9 This is because $y_i = \tilde{y}_{n-i+1}$ and $\tilde{y}_i = y_{n-i+1}$.
10 The Gini SEF, as a rank dependent expected utility function, also draws some attention in economic theory; see, for example, Chew, Karni, and Safra (1987), Quiggin (1982), Segal and Spivak (1990), and Yaari (1987).
11 The two equations are identical because $\bar{\mathbf{y}} = \bar{\tilde{\mathbf{y}}}$, $y_i = \tilde{y}_{n-i+1}$, and $\tilde{y}_i = y_{n-i+1}$.

(1983), respectively. Since these indices (BD and C indices) are defined using an EDEI corresponding to a specific SEF, they permit a direct link between the S and SST indices to their underlying SEF.

Consistent with the S index, the BD index focuses on the incomes of the poor $\mathbf{y}_p$ or the truncated income distribution by excluding the non-poor. The BD index is defined as

$$I_{BD}(\mathbf{y}_p) = H\left[\frac{z - \Xi(\mathbf{y}_p)}{z}\right], \tag{6}$$

where Ξ is the EDEI function of $\mathbf{y}_p$ for some increasing and strict S-concave SEF.[12] In the definition, the EDEI function is generic. The S index is defined in Sen (1976) as

$$I_S(\mathbf{y}_p) = H[\bar{\mathbf{x}}_p + (1 - \bar{\mathbf{x}}_p)G(\mathbf{y}_p)]. \tag{7}$$

The BD index $I_{BD}(\mathbf{y}_p)$ with the Gini EDEI $\Xi_G(\mathbf{y}_p)$ is the S index; that is,

$$I_S(\mathbf{y}_p) = I^G_{BD}(\mathbf{y}_p) = H\left[\frac{z - \Xi_G(\mathbf{y}_p)}{z}\right] = H\Xi_G(\mathbf{x}_p). \tag{8}$$

(see Blackorby and Donaldson 1980, 1054–5). Equation (8) provides a mathematical structure based on which one can see why the S index is explicitly related to the underlying Gini SEF.[13]

Following the idea of Thon (1979) and Takayama (1979), Chakravarty (1983) proposed the C index for the censored income vector $\overset{*}{\mathbf{y}}$:

$$I_C(\overset{*}{\mathbf{y}}) = \frac{z - \Xi(\overset{*}{\mathbf{y}})}{z}, \tag{9}$$

where Ξ is the EDEI function for some increasing and strict S-concave SEF. Note that the EDEI is generic. The SST index of poverty intensity is defined in Shorrocks (1995) as either

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)x_i \tag{10}$$

12 A function $f: \mathbf{R}^n_+ \to \mathbf{R}^1$ is S-concave if $f(\mathbf{B}\mathbf{y}) \geq f(\mathbf{y})$ for all $\mathbf{y} \in \mathbf{R}^n_+$, where $\mathbf{B}$ is a bistochastic matrix, a square matrix of order n with all elements being non-negative and the elements in each row/column being summed up to one (i.e., for elements b_{ij}s of $\mathbf{B}$, $\sum_{j=1}^n b_{ij} = 1$ for $i = 1,2,\ldots,n; \sum_{i=1}^n b_{ij} = 1$ for $j = 1,2,\ldots,n$).

13 It should be noted that Sen (1976) started from Axioms R (Ordinal Rank Weights), M (Monotonic Welfare), and N (Normalized Poverty Value), which have Gini social welfare implications.

or

$$I_{SST}(\mathbf{y}_p) = \frac{1}{n^2}\sum_{i=1}^{q}(2n - 2i + 1)x_i. \tag{11}$$

The C index $I_C(\overset{*}{\mathbf{y}})$ with the Gini EDEI $\Xi_G(\overset{*}{\mathbf{y}})$ is the SST index $I_{SST}(\overset{*}{\mathbf{y}})$; that is,

$$I_{SST}(\overset{*}{\mathbf{y}}) = I_C^G(\overset{*}{\mathbf{y}}) = \frac{z - \Xi_G(\overset{*}{\mathbf{y}})}{z} = \Xi_G(\mathbf{x}) \tag{12}$$

(see Chakravarty 1997). Equation (12) provides a mathematical structure based on which one can see why the SST index is explicitly related to the Gini SEF.

From the above discussion, it becomes clear that a higher (lower) value of the poverty intensity measured by either I_S or I_{SST} means that the lower (higher) level of social welfare measured by the Gini SEF.

3.2. Common multiplicative decomposition

Neither the S index nor the SST index permits additive decomposition, although they possess desirable properties. The following propositions show that both S and SST indices permit a common multiplicative decomposition into the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios.

PROPOSITION 1. *The S index has the following multiplicative decomposition:*

$$I_S(\mathbf{y}_p) = H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}}_p)), \tag{13}$$

where $\tilde{\mathbf{x}}_p$ *has elements in non-decreasing order.*

Proof. Using the relationship between G and Ξ_G (equation (3)) and equation (5), rewrite equation (8) as

$$I_S(\mathbf{y}_p) = H\Xi_G(\mathbf{x}_p) = H\bar{\mathbf{x}}_p(1 - G(\mathbf{x}_p)) = H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}}_p)), \tag{14}$$

where $\mathbf{x}_p$ ($\tilde{\mathbf{x}}_p$) has elements in non-increasing (non-decreasing) order. Q.E.D.

As can be seen from the above proposition, we can view the S index as the product of the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios of the *poor*.

It is also interesting to compare the multiplicative decomposition of the original S index with the one presented here. As Sen (1976) pointed out, for a large q, the S index is defined as in equation (7), where the Gini index is for incomes of the poor. We show that the S index can be written, alternatively, as in equation (13), where the Gini index is for poverty gap ratios of the poor. Equation (13) is a bit simpler than equation (7) and permits a simpler geometric interpretation, as is shown later in this paper.

The following proposition states that the SST index permits similar multiplicative decomposition.

PROPOSITION 2. *The SST index has the following multiplicative decomposition:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}})), \tag{15}$$

where $\tilde{\mathbf{x}}$ has elements in non-decreasing order.

Proof. Using the relationship between G and Ξ_G (equation (3)), equation (5) and $\bar{\mathbf{x}} = H\bar{\mathbf{x}}_p$ rewrite equation (12) as

$$I_{SST}(\overset{*}{\mathbf{y}}) = \Xi_G(\mathbf{x}) = \bar{\mathbf{x}}(1 - G(\mathbf{x})) = H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}})), \tag{16}$$

where $\mathbf{x}$ ($\tilde{\mathbf{x}}$) has elements in non-increasing (non-decreasing) order. Q.E.D.

As can be seen from the above proposition, we can express the SST index as the product of the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios of the *population*.

Note that the two indices differ only by the argument of $G(\cdot)$. The S index has a component $G(\tilde{\mathbf{x}}_p)$ while the SST index has a component $G(\tilde{\mathbf{x}})$. Since the poverty gap ratios of the non-poor subpopulation are zeros and the poor and the non-poor subpopulations do not overlap in the censored income vector $\overset{*}{\mathbf{y}}$, the Gini index of poverty gap ratios of the *population* can be decomposed into two components as follows:

LEMMA 1. *The Gini index of poverty gap ratios of the population, $G(\tilde{\mathbf{x}})$, is the sum of the Gini index of the average poverty gap ratios between the non-poor and the poor subpopulations, $(1 - H)$, and the poverty-rate-weighted Gini index of poverty gap ratio of the poor, $HG(\tilde{\mathbf{x}}_p)$, as follows:*

$$G(\tilde{\mathbf{x}}) = (1 - H) + HG(\tilde{\mathbf{x}}_p). \tag{17}$$

Proof. From equation (3), we have

$$G(\mathbf{x}) = 1 - \frac{1}{n^2\bar{\mathbf{x}}}\sum_{i=1}^{n}(2n - 2i + 1)x_i, \tag{18}$$

where the poverty gap ratio vector of the population, $\mathbf{x}$, has elements in non-increasing order (i.e., the poor subpopulation takes the top partition of the column vector, while the non-poor subpopulation takes the bottom partition). Similarly, from equation (3), we have

$$G(\mathbf{x}_p) = 1 - \frac{1}{q^2\bar{\mathbf{x}}_p}\sum_{i=1}^{q}(2q - 2i + 1)x_i, \tag{19}$$

where the poverty gap ratio vector of the poor, $\mathbf{x}_p$, has elements in non-increasing order. It is known that $\bar{\mathbf{x}} = H\bar{\mathbf{x}}_p$. From equation (18) we get

$$G(\mathbf{x}) = 1 - \frac{q}{n}\frac{1}{q^2\bar{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)x_i - 2\left(1-\frac{q}{n}\right). \tag{20}$$

It can be further rewritten as

$$G(\mathbf{x}) = \frac{q}{n}\left\{1 - \frac{1}{q^2\bar{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)x_i\right\} - \left(1-\frac{q}{n}\right). \tag{21}$$

Thus,

$$G(\mathbf{x}) = (H-1) + HG(\mathbf{x}_p). \tag{22}$$

Applying equation (5) to G on the left-hand-side of equation (22), equation (22) becomes

$$G(\tilde{\mathbf{x}}) = -(H-1) - HG(\mathbf{x}_p). \tag{23}$$

Applying equation (5) to G on the right-hand-side of equation (23), equation (23) becomes

$$G(\tilde{\mathbf{x}}) = (1-H) + HG(\tilde{\mathbf{x}}_p). \tag{24}$$

Q.E.D.

PROPOSITION 3. *The SST index and the S index are related in the following way:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = HI_S(\mathbf{y}_p) + 2H(1-H)\bar{\mathbf{x}}_p. \tag{25}$$

Proof. Combining the results in lemma 1 and proposition 2 gives

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\bar{\mathbf{x}}_p(2(1-H) + H(1+G(\tilde{\mathbf{x}}_p))). \tag{26}$$

Further manipulation of equation (26) gives equation (25). Zheng (1997) stated the same result (equation (3.9), p. 146) without giving the details of the proof. Q.E.D.

According to Chakravarty (1990, theorem 6.9), if the SEF is completely strictly recursive, then

$$I_{BD}(\mathbf{y}_p) < I_C(\overset{*}{\mathbf{y}}). \tag{27}$$

In other words, the BD index is bounded above by the C index. For the Gini SEF, which is the underlying SEF for the S and SST indices but not completely strict recursive, the similar relationship holds.

PROPOSITION 4. *The S index is bounded above by the SST index, that is,*

$$I_S(\mathbf{y}_p) < I_{SST}(\overset{*}{\mathbf{y}}). \tag{28}$$

Proof. Based on equation (25), we have

$$I_{SST}(\overset{*}{\mathbf{y}}) = HI_S(\mathbf{y}_p) + 2H(1-H)\bar{\mathbf{x}}_p.$$

Given $2H(1-H)\bar{\mathbf{x}}_p > 0$ and $H < 1$,

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) &< HI_S(\mathbf{y}_p) \\ &< I_S(\mathbf{y}_p). \end{aligned}$$

"<" should be ">" here till the end of the proof.

Q.E.D.

The common multiplicative decomposability of the S and SST indices allows economists to evaluate social welfare measured by the poverty *intensity* and its contributing components. The multiplicative decomposition of the S and SST indices can be transformed, through the logarithmic transformation, to be additive in a simple form. In the following corollary, we use I for either the S index or the SST index and G for the Gini index of poverty gap ratios of either the poor or the population.

COROLLARY 1. *Since the S and SST indices of poverty intensity take the form of*

$$I = H\bar{\mathbf{x}}_p(1+G), \tag{29}$$

then

$$\Delta I = \Delta H + \Delta\bar{\mathbf{x}}_p + \Delta(1+G), \tag{30}$$

where $\Delta x = \ln x_t - \ln x_{t-1} \approx (x_t - x_{t-1})/x_{t-1}$ *approximates the percentage change in* x *for a small change in* x.

Depending on the purpose of research, one may use the same poverty line $z_t = z_{t-1} = z$ for I_t and I_{t-1} and their components or different poverty lines z_t and z_{t-1}, respectively, for I_t and I_{t-1} and their components.

Osberg and Xu (1997, 1999, 2000) applied this multiplicative decomposition for the SST index to the international and regional comparative studies. Economists at Statistics Canada have also adopted this methodology to analyse low-income intensity among Canadian children (Myles and Picot 2000).

The common multiplicative decomposition also allows policy makers to use three specific anti-poverty policy 'targets' (rate, gap, and inequality) in reducing

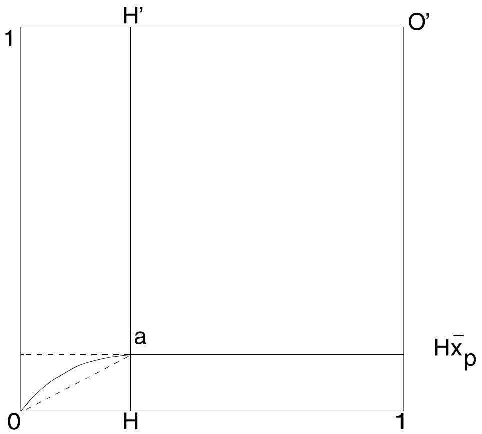

FIGURE 1 Deprivation profile

poverty intensity. These targets may be used to monitor the effectiveness of the anti-poverty policy.[14]

3.3. Similar geometric interpretations

The S index permits a simpler geometric interpretation that is somewhat different from that of Sen (1976) but is quite close to that of the SST index proposed by Shorrocks (1995). For comparison purposes, we also present and interpret the SST index geometrically.

Note again that the relative deprivation measure is $x_i = (z - y_i)/z$ if $z > y_i$, $x_i = 0$ otherwise, for $i = 1,2,\ldots,n$. The x_is are in non-increasing order. The first q x_is are positive for the poor who are deprived, and the rest are zeros for the non-poor.

The deprivation profile can be graphed by plotting $1/n \sum_{i=1}^{r} x_i$ against r/n for $r = 1,2,\ldots,n$ in a unit box. As shown in figure 1, the poverty profile starts from the origin, reaches out concavely to the point a and then becomes horizontal from the point a to the point $H\bar{\mathbf{x}}_p$. The point H represents the poverty rate, and the point $H\bar{\mathbf{x}}_p$ represents the average poverty gap ratio of the population, $\bar{\mathbf{x}}$. Since the deprivation measures $\{x_i\}$ are in non-increasing order, the concave arc $0a$ is in fact an inverted

14 Bourguignon and Fields (1997) and Ravallion, van de Walle, and Gautam (1995) discussed the relationship between poverty measures and anti-poverty policy actions. As Bourguignon and Fields (1997) noted, if there is a qualitative difference (e.g., in functions) between being poor or non-poor, the poverty rate is of specific interest. Similarly, if the depth of poverty is of a major social concern, the average poverty gap ratio is of specific interest. If the dispersion of deprivations demands more social attention, inequality of deprivations is clearly of greater importance. In practice, however, the changes in inequality of deprivations over time or across jurisdictions, relative to those in the poverty rate or average poverty gap ratio, are of much smaller magnitude. See Osberg and Xu (1997, 2000).

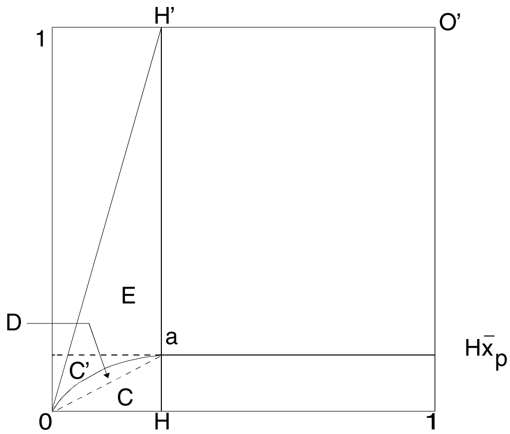

FIGURE 2 Geometric interpretation of the S index

generalized Lorenz curve for the deprivation measures $\{x_i\}$, which represents the inequality of poverty gap ratios of the poor. The dotted straight line linking the origin 0 and the point a would be a segment of the poverty profile if the poor had identical incomes (i.e., their poverty gap ratios were all identical). Since the non-poor have zero poverty gap ratios, the horizontal segment from the point a to the point $H\bar{\mathbf{x}}_p$ of the deprivation profile has no significant information but shows the non-poor account for the $1 - H$ proportion of the population.

In figure 2, we show the S index has a simple geometric interpretation that is similar to that of the Gini index. Note that triangle $0H'H$ is Area E. Triangle $0Ha$ is Area C. The space between arc $0a$ and the dotted straight line linking the origin 0 and the point a is Area D. Thus,

$$\text{Area } E = \frac{1}{2} H \tag{31}$$

$$\text{Area } C = \frac{1}{2} H^2 \bar{\mathbf{x}}_p. \tag{32}$$

Area D can be computed from the fact that the Gini index of poverty gap ratios of the poor is given by[15]

$$G(\tilde{\mathbf{x}}_p) = \frac{\text{Area } D}{\text{Area } C'} = \frac{\text{Area } D}{\text{Area } C}. \tag{33}$$

15 Note that Area C', the triangle formed by two dotted straight lines and the vertical axis, is identical to Area C. Also note that the Gini index of poverty gap ratios of the poor ($\tilde{\mathbf{x}}_p$) here is defined as the ratio of two areas, in the rectangle of the length H and the height $\bar{\mathbf{x}} = H\bar{\mathbf{x}}_p$ in the larger unit box. This rectangle can be transformed into a unit box by rescaling the vertical and horizontal axes without affecting the ratio of the two areas, and hence the Gini index of poverty gap ratios of the poor is defined on the ratio of the two areas.

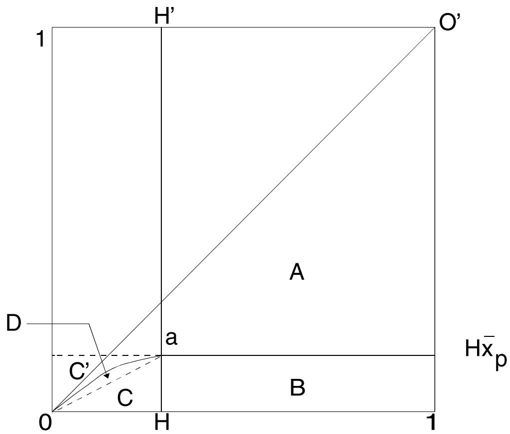

FIGURE 3 Geometric interpretation of the SST index

Using equations (32) and (33) yields

$$\text{Area } D = \text{Area } C \times G(\tilde{\mathbf{x}}_p) = \frac{1}{2} H^2 \bar{\mathbf{x}}_p G(\tilde{\mathbf{x}}_p).$$

The S index is simply the ratio of the sum of Areas C and D to Area E; that is,

$$\begin{aligned} I_S(\mathbf{y}_p) &= \frac{\text{Area } C + \text{Area } D}{\text{Area } E} \\ &= \frac{\frac{1}{2} H^2 \bar{\mathbf{x}}_p + \frac{1}{2} H^2 \bar{\mathbf{x}}_p G(\tilde{\mathbf{x}}_p)}{\frac{1}{2} H} \qquad (34) \\ &= H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}}_p)). \end{aligned}$$

For a better understanding of the common multiplicative decomposition and similar geometric interpretations, we also analyse the geometric interpretation of the SST index in a similar fashion in figure 3. Let the lower triangle of the unit box in figure 3 be Area A and the rectangle at the lower right-hand corner of the unit box be Area B. Thus

$$\text{Area } A = \frac{1}{2} \qquad (35)$$

and

$$\text{Area } B = (1 - H)H\bar{\mathbf{x}}_p = H\bar{\mathbf{x}}_p - H^2 \bar{\mathbf{x}}_p. \qquad (36)$$

According to equation (15), the SST index can be expressed as

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\bar{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}})). \tag{37}$$

Further, using equations (17), equation (37) becomes

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\bar{\mathbf{x}}_p(2 - H + HG(\tilde{\mathbf{x}}_p)). \tag{38}$$

The SST index is the ratio of the sum of Areas B, C, and D to Area A; that is,

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) &= \frac{\text{Area } B + \text{Area } C + \text{Area } D}{\text{Area } A} \\ &= \frac{H\bar{\mathbf{x}}_p\left[(1-H) + \frac{1}{2}H + \frac{1}{2}HG(\tilde{\mathbf{x}}_p)\right]}{\frac{1}{2}} \\ &= H\bar{\mathbf{x}}_p(2 - H + HG(\tilde{\mathbf{x}}_p)). \end{aligned} \tag{39}$$

The similar geometric interpretation puts both S and SST indices in a Gini-like framework which shows clearly that H, $\bar{\mathbf{x}}_p$, and G are three key components determining the poverty intensity. For applied economists and policy analysts, this graphical approach can effectively convey information about poverty (see Xu and Osberg 2001b for an example).

4. Concluding remarks

In this paper we have discussed the common underlying social evaluation function for the Sen indices of poverty intensity and presented a unified multiplicative decomposition framework for the Sen indices.

The Sen indices (the S and SST indices) share a common Gini social evaluation function and have a common multiplicative decomposition structure – being the product of the poverty rate, average poverty gap ratio of the poor, and one plus the Gini index of the poverty gap ratios as follows:

$$\begin{pmatrix}\text{The S}\\ \text{index}\end{pmatrix} = \begin{pmatrix}\text{poverty}\\ \text{rate}\end{pmatrix} \times \begin{pmatrix}\text{average}\\ \text{poverty}\\ \text{gap}\\ \text{ratio}\end{pmatrix} \times \begin{pmatrix}1 + \text{Gini index}\\ \text{of poverty gaps}\\ \text{of the poor}\end{pmatrix}$$

and

$$\begin{pmatrix}\text{The SST}\\ \text{index}\end{pmatrix} = \begin{pmatrix}\text{poverty}\\ \text{rate}\end{pmatrix} \times \begin{pmatrix}\text{average}\\ \text{poverty}\\ \text{gap}\\ \text{ratio}\end{pmatrix} \times \begin{pmatrix}1 + \text{Gini index}\\ \text{of poverty gaps}\\ \text{of the population}\end{pmatrix}.$$

This common multiplicative decomposition structure (1) gives the two indices a much more straightforward interpretation of poverty intensity, (2) allows the indices to be computed much more easily via commonly known poverty measures (the poverty rate and average poverty gap ratio) and inequality measures (the Gini index of the poverty gap ratios), and (3) permits the indices to have the Gini-index-like geometric interpretations. The practical implication of the multiplicative decomposition is that the Sen indices can be linearized so that the percentage change in these indices are additively decomposable.

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