On Sen's Approach to Poverty Measures and Recent Developments

Kuan Xu and Lars Osberg*

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Abstract

In this paper we discuss the axiomatic approach to poverty measures and propose a unified framework for the Sen indices of poverty intensity which shows an explicit connection between the indices and their common underlying social evaluation function. We also identify the common multiplicative decomposition of the indices that allows simple and similar geometric interpretations and easy numerical computation. These results are easy to understand and useful to policy makers in both developed and developing countries.
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1 Introduction

How much poverty is there?

Is poverty increasing or decreasing?

To answer such seemingly simple questions, analysts must choose (1) a criterion for deciding whether an individual is poor (such as whether his or her income is below the poverty line) and (2) an index which summarizes the amount of poverty in a society. This paper concerns the second issue, building on the contribution of Sen (1976).

Some indices of poverty (such as the poverty rate) are easy to understand but often misleading. Some others have desirable ethical properties but are rarely used in policy debates because of their complexity. When proposing an index of poverty, researchers should therefore avoid both the danger that the proposed measure will be theoretically unsound and the hazard that the measure will be so complex as to be not understandable by policy makers, and hence never used. The advantage of the Sen family of poverty indices is that they can be justified at both the theoretical level of ethical soundness and the practical level of easy communicability to the general public.

In 1976, Sen proposed both an axiomatic approach to poverty research and a specific index. Since then, poverty measurement has become an active research agenda and a vast theoretical literature has developed.[1] Applications of the Sen index (or S index [2]) and the modified Sen index (or SST index[3]) have appeared in empirical poverty studies by Bishop, Formby and Zheng (1997), Myles and Picot (2000), Osberg (2000), Osberg and Xu (1997, 1999, 2000), Rongve (1997), and Xu (1998) among others.

The Sen indices are based on a set of well justified and commonly agreed axioms. But it is also desirable to understand the meaning of the Sen indices

[1]See Zheng (1997) and Foster and Sen (1997), and the references therein, for two recent comprehensive surveys.

[2]See Sen (1976). The index is called the S index in Sen (1997).

[3]The index is called the modified Sen index in Shorrocks (1995) and Sen (1997). Shorrocks (1995) proposed the index. Zheng (1997) noted that the modified Sen index is identical to the limit of Thon's modified Sen index [Thon (1979, 1983)]. Thus, we call it the Sen-Shorrocks-Thon index [see Osberg and Xu (1997, 1999, 2000)].

in terms of social welfare evaluation. The social evaluation function that the Sen indices jointly share has not yet been explicitly summarized in the literature.[4] Bourguignon and Fields (1997) pointed out that poverty measures can be interpreted as gauging the social welfare losses when persons have low incomes. Blackorby and Donaldson (1978, 1980) and Chakravarty (1983, 1997) laid a solid ground for interpreting the social welfare meaning of many inequality and poverty indices. We use their work to examine the Sen indices' underlying social welfare function.

To make the Sen indices more understandable to policy makers, it is also desirable to examine the relationship between the Sen indices and their contributing components [see, for example, and Birdsall and Londono (1997), and Phipps (1999) among others]. *Additive* decomposability is a property of the Foster, Greer and Thorbecke (1984) index of poverty.[5] While the Sen indices (also their generalizations such as BD index[6] and C index[7]) do not have this property in general,[8] they do have the property of *multiplicative* decomposability.[9] Bourguigon and Fields (1997) noted that some additive poverty measures may bias anti-poverty policy to paying attention to the richest of the poor and that an appropriate poverty measure should facilitate a comprehensive evaluation of anti-poverty policy actions. Since the Sen indices have desirable ethical properties and are multiplicatively decomposable, they can be readily used to measure the multidimensional impacts of anti-poverty policy actions.

In the literature on income inequality, the Gini index is perhaps the mostly used index of inequality, partly because it has a useful and intuitive geometric interpretation. The SST index also has a useful and intuitive geometric interpretation [see Shorrocks (1995), Jenkins and Lambert (1997), Osberg and Xu (1997, 2000), and Xu and Osberg (1998)]. But the geometric interpretation of the S index is less intuitive [see Sen (1976, p. 226)]. The

[4]Dalton (1920) in his pioneering paper suggested that any measure of income inequality has an underlying social welfare function. This has been made precise by Kolm (1969), Atkinson (1970), and Sen (1973).

[5]See Chapter 7 of Chakravarty (1990) for a detailed survey of the additive decomposition of the poverty intensity indices.

[6]See Blackorby and Donaldson (1980).

[7]See Chakravarty (1983).

[8]The Chakravarty index with the symmetric means of order r ($r < 1$) is an exception.

[9]This is first briefly mentioned for the S index by Clark, Hemming, and Ulph (1981) for the S index and then examined for the SST index by Osberg and Xu (1997, 1999, 2000).

common multiplicative decomposability of the Sen indices suggests that the decompositions must have similar useful and intuitive geometric interpretations.

In this paper, we examine the common underlying social evaluation function, multiplicative decomposability, and geometric interpretations of the Sen indices. We show that: (1) the Sen indices are based on a set of well justified and commonly agreed axioms; (2) the Sen indices share a common Gini social evaluation function, in which the social welfare is evaluated as the rank-weighted "average" of incomes; (3) the Sen indices also have a common multiplicative decomposition structure; each index can be expressed as the product of the poverty rate, average poverty gap ratio[10] of the poor and one plus the Gini index of the poverty gap ratios; (4) the SST index is a linear transformation of the S index and vice versa; (5) the common multiplicative decomposability of the Sen indices permits similar useful and intuitive geometric interpretations, renders them easy to understand and compute, and allows further subgroup decompositions; and (6) because of the common multiplicative decomposability, the Sen indices can be linearized so that they are additively decomposable —a useful result for empirical comparisons and policy analysis.

In section 2, the notation, some basic concepts and axioms, and historical background of the Sen's contributions are discussed. In section 3, we discuss the common underlying social evaluation function, the common multiplicative decomposability, useful and intuitive geometric interpretations of the Sen indices. We will illustrate an empirical example for applying the one of the Sen's poverty indices in section 4. Finally, section 5 concludes.

2 The Sen Family of Poverty Indices

2.1 Notation

Let $\mathbf{y} = [y_1, y_2, \dots, y_n]^\top$ be the income vector of a population of size n with (individual or family) incomes sorted in non-decreasing order, where "$\top$" is used to denote transposition of a matrix or a vector. Let $\widetilde{\mathbf{y}}$ be $\mathbf{y}$ with incomes sorted in non-increasing order where the notation " $\sim$ " is used for sorting a vector $\mathbf{x}$ in opposite order. Let the poverty line be $z > 0$. Let the

[10]Sometimes, the average poverty gap ratio is simply called the poverty gap.

number of the poor be q. Hence the poverty rate H is $\frac{q}{n}$. A censored income vector is obtained by setting $\overset{*}{y}_i = y_i$ if $y_i < z$ and $\overset{*}{y}_i = z$ otherwise,[11] that is $\overset{*}{\mathbf{y}} = [\overset{*}{y}_1, \overset{*}{y}_2, \ldots, \overset{*}{y}_n]^\top$. The income vector of the poor, $\mathbf{y}_p = [y_1, y_2, \ldots, y_q]^\top$, is a truncated income vector generated from $\overset{*}{\mathbf{y}}$ by deleting z's. The average of an income vector $\mathbf{y}$ is given by $\overline{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

The poverty gap ratio vector of the population is defined as $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$ where the poor have poverty gap ratios $x_i = \frac{z - y_i}{z}$, $i = 1, 2, \ldots, q$, and the non-poor have zero poverty gap ratios. Similarly, the poverty gap ratio vector of the poor is given by $\mathbf{x}_p = [x_1, x_2, \ldots, x_q]^\top$ where the poor have poverty gap ratios $x_i = \frac{z - y_i}{z}$, $i = 1, 2, \ldots, q$, and the non-poor's zero poverty gap ratios are excluded. Please note the elements in both $\mathbf{x}$ and $\mathbf{x}_p$ are in non-increasing order. The average poverty gap ratio of the population (the poor) is denoted by $\overline{\mathbf{x}}$ ($\overline{\mathbf{x}}_p$) and $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$.

2.2 The Axiomatic Approach

Amartya Sen is well-known as an advocate of the axiomatic approach to poverty measures or indices. Prior to his proposal, commonly used poverty measures were often advanced on an ad hoc basis, but he argued that poverty measures should be consistent with a set of ethically defensible criteria. The literature which has followed Sen's contribution has established the following basic criteria, more formally known as axioms:[12]

(1) **Focus Axiom**: The poverty index should be independent of the non-poor population.

(2) **Weak Monotonicity Axiom**: A reduction in a poor person's income, holding other incomes constant, must increase the poverty index.

(3) **Impartiality Axiom**: The poverty index may be defined over ordered income profiles without loss of generality.

(4) **Weak Transfer Axiom**: An increase in the poverty index should occur if the poorer of the two individuals involved in an upward transfer of income is poor and if the set of the poor people does not change.

(5) **Strong Upward Transfer Axiom**: An increase in the poverty index should occur if the poorer of the two individuals involved in an upward

[11] We use the weak definition of the poor here—a poor person's income is less than the poverty line—as it is generally treated in the literature.

[12] See Chakravarty (1990) for more detailed discussion.

transfer of income is poor.[13]

(6) **Continuity Axiom**: The poverty index must vary continuously with incomes.

(7) **Replication Invariance Axiom**: The poverty index does not change if it is computed based on an income distribution that is generated by the k-fold replication of an original income distribution.

Although these axioms are ethically agreeable, not all commonly used poverty measures satisfy them, in particular the first four fundamental axioms.

The poverty rate ($H = \frac{q}{n}$) is the most commonly used poverty measure, and is defined as the percentage of the population whose incomes are under the poverty line. This measure satisfies the focus axiom. That is, if more (less) individual incomes fall below the poverty line, the number of the poor increases (decreases) and the poverty rate will increase (decrease). But the poverty rate violates both the weak monotonicity and weak transfer axioms, because it is unaffected by how far and how unevenly the individual incomes of the poor fall below the poverty line. In other words, it does not reflect changes in the extent of the shortfall of income from the poverty line and is completely insensitive to the distribution of income among the poor [see Sen (1976)]. In practice, this means that if an anti-poverty policy were to be designed to reduce the poverty rate, the easiest approach would be to subsidize the richest of the poor with just barely enough additional income to lift him or her out of poverty. Clearly, this would be a controversial policy action.

The average poverty gap ratio of the poor ($\overline{\mathbf{x}}_p = \frac{1}{q}\sum_{i=1}^{q}\frac{z-y_i}{z}$) is another commonly used index. It measures the average shortfall of income from the poverty line relative to the poverty line. If the income shortfalls from the poverty line relative to the poverty line, on average, are larger (smaller), the average poverty gap ratio will increase (decrease). However, it is unaffected by changes in the percentage of the population who are poor. Furthermore, this measure violates the weak transfer axiom because it is insensitive to the distribution of income among the poor [see Sen (1976)]. An income transfer from one poor person to another poor person without lifting any of the two

[13]Note that the strong upward transfer axiom implies the weak transfer axiom since the strong upward transfer axiom allows the poor sub-population to be either the same or to change while the weak transfer axiom does not permit the poor sub-population to change.

out of poverty will not change the average poverty gap ratio.

These dissatisfactions of the commonly used poverty measures led Sen (1976) to propose the S index based on the focus, monotonicity, and weak transfer axioms. The original version of the S index is derived from these axioms as a function of the income distribution of the poor $\mathbf{y}_p$:

$$I_{So}(\mathbf{y}_p) = H\left[1 - (1 - \overline{\mathbf{x}}_p)\left(1 - G(\mathbf{y}_p)\left(\frac{q}{1+q}\right)\right)\right] \tag{1}$$

where $G(\mathbf{y}_p)$ is the Gini index of the income distribution of the poor. Unfortunately the original version of the S index does not satisfy the strong upward transfer, continuity, and replication invariance axioms.[14] Sen (1976) derived the second version of the S index from equation (1) assuming q is large so that the term $q/(1+q)$ goes to 1:

$$I_S(\mathbf{y}_p) = H\left[\overline{\mathbf{x}}_p + (1 - \overline{\mathbf{x}}_p)\, G(\mathbf{y}_p)\right]. \tag{2}$$

This second version of the S index, which is called the S index in the empirical literature, satisfies the replication invariance axiom.

Given that the original version of the S index $I_{So}(\mathbf{y}_p)$ does not satisfy several axioms as mentioned above, Shorrocks (1995) proposed the modified Sen index or the SST index and showed that the SST index satisfies the strong upward transfer, continuity, and replication invariance axioms. The SST index is defined as a function of the censored income distribution $\overset{*}{\mathbf{y}}$:

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)x_i. \tag{3}$$

Osberg and Xu (1997) found that the SST index can be viewed as a product of three commonly used poverty and inequality measures: the poverty rate, average poverty gap ratio, and one plus the Gini index of the poverty gap ratios of the population. This simplifies the understanding and use of the SST index. Osberg and Xu (1997, 1999, 2000) applied this multiplicative decomposition for the SST index to international and regional comparative studies. Economists at Statistics Canada have also adopted this methodology to analyze low-income intensity among Canadian children [Myles and Picot (2000)].

[14] See Shorrocks (1995, p. 1225) and Sen (1997, p. 171).

In the remaining part of the paper, we will show the S and SST indices in fact share an identical social welfare or evaluation function, have the same decomposition structure, and possess similar Gini-index-like geometric interpretations.

3 Common SEF and Multiplicative Decomposition

3.1 Common Gini Social Evaluation Function

To analyze the social welfare implication of the Sen indices, we need to utilize the concept of the equally distributed equivalent income (EDEI), or the representative income proposed by Atkinson (1970), Kolm (1969), and Sen (1973). For a particular social evaluation function (SEF), an EDEI given to every individual could be viewed as identical in terms of social welfare to an actual income distribution. Let $W(\mathbf{y}) = \phi(\overline{W}(\mathbf{y}))$ be a homothetic (ordinal) SEF of income with ϕ being an increasing function and $\overline{W}$ being a linearly homogeneous function. Let ξ be the EDEI and $\mathbf{1}$ be a column vector of ones with an appropriate dimension. Then, $W(\xi \cdot \mathbf{1}) = W(\mathbf{y})$ or $\overline{W}(\xi \cdot \mathbf{1}) = \overline{W}(\mathbf{y})$. Given that $\overline{W}$ is positively linearly homogeneous, EDEI is computed by $\xi = \frac{\overline{W}(\mathbf{y})}{\overline{W}(\mathbf{1})} = \Xi(\mathbf{y})$. The SEF ($W$) and the EDEI ($\Xi$) have an one-to-one corresponding relationship.

For example, the Gini SEF is $\overline{W}_G(\mathbf{y}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)y_i$.[15] Its corresponding EDEI function is

$$\Xi_G(\mathbf{y}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)y_i \tag{4}$$

or

$$\Xi_{\widetilde{G}}(\widetilde{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2i - 1)\widetilde{y}_i \tag{5}$$

[15] This is because

$$\overline{W}_G(\mathbf{1}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1) = 1.$$

with $\Xi_G(\mathbf{y}) = \Xi_{\widetilde{G}}(\tilde{\mathbf{y}})$.[16] The Gini SEF attaches a higher weight to a lower level of income and vice versa. The weight is determined by the rank of an income rather than the size of the income.[17]

The Gini index can be defined in terms of the Gini EDEI and the mean income as

$$G(\mathbf{y}) = 1 - \frac{\Xi_G(\mathbf{y})}{\overline{\mathbf{y}}} = 1 - \frac{1}{n^2\overline{\mathbf{y}}}\sum_{i=1}^{n}(2n-2i+1)y_i \tag{6}$$

or

$$\widetilde{G}(\tilde{\mathbf{y}}) = 1 - \frac{\Xi_{\widetilde{G}}(\tilde{\mathbf{y}})}{\overline{\mathbf{y}}} = 1 - \frac{1}{n^2\overline{\mathbf{y}}}\sum_{i=1}^{n}(2i-1)\tilde{y}_i, \tag{7}$$

where $\mathbf{y}$ ($\tilde{\mathbf{y}}$) has elements in non-decreasing (non-increasing) order.[18] Note that $G(\mathbf{y}) = \widetilde{G}(\tilde{\mathbf{y}})$ in equations (6) and (7) are identical but $G(\cdot)$ and $\widetilde{G}(\cdot)$ have different functional forms and the elements in $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are sorted differently. Also note that[19]

$$G(\mathbf{y}) = -G(\tilde{\mathbf{y}}). \tag{8}$$

The link between the S and SST indices can be better understood based on the BD and C indices introduced by Blackorby and Donaldson (1980) and Chakravarty (1983), respectively. Since these indices (BD and C indices) are defined using an EDEI corresponding to a specific SEF, they permit a direct link between the S and SST indices to their underlying SEF.

Consistent with the S index, the BD index focuses on the incomes of the poor $\mathbf{y}_p$ or the truncated income distribution by excluding the non-poor population. The BD index is defined as:

$$I_{BD}(\mathbf{y}_p) = H\left[\frac{z - \Xi(\mathbf{y}_p)}{z}\right] \tag{9}$$

where Ξ is the EDEI function of $\mathbf{y}_p$ for some increasing and strict S-concave

[16]This is because $y_i = \tilde{y}_{n-i+1}$ and $\tilde{y}_i = y_{n-i+1}$.

[17]The Gini SEF, as a rank dependent expected utility function, also draws some attention in economic theory; see, for example, Chew and Safra (1987), Quiggin (1982), Segal and Spivak (1990), and Yaari (1987).

[18]The two equations are identical because $\overline{\mathbf{y}} = \overline{\tilde{\mathbf{y}}}$, $y_i = \tilde{y}_{n-i+1}$, and $\tilde{y}_i = y_{n-i+1}$.

[19]See Fei, Ranis, and Kuo (1978) and Xu and Osberg (2001).

SEF.[20] In the definition, the EDEI function is generic. The S index is defined as

$$I_S(\mathbf{y}_p) = H\left[\overline{\mathbf{x}}_p + (1 - \overline{\mathbf{x}}_p)\, G(\mathbf{y}_p)\right]. \tag{10}$$

The BD index $I_{BD}(\mathbf{y}_p)$ with the Gini EDEI $\Xi_G(\mathbf{y}_p)$ is the S index; that is

$$I_S(\mathbf{y}_p) = I^G_{BD}(\mathbf{y}_p) = H\left[\frac{z - \Xi_G(\mathbf{y}_p)}{z}\right] = H\Xi_G(\mathbf{x}_p) \tag{11}$$

[see Blackorby and Donaldson (1980, pp. 1054–1055)]. Equation (11) provides a mathematical structure based on which one can see why the S index is explicitly related to the underlying Gini SEF.[21]

Following the idea of Thon (1979) and Takayama (1979), Chakravarty (1983) proposed the C index for the censored income vector $\overset{*}{\mathbf{y}}$:

$$I_C(\overset{*}{\mathbf{y}}) = \frac{z - \Xi(\overset{*}{\mathbf{y}})}{z}, \tag{12}$$

where Ξ is the EDEI function for some increasing and strict S-concave SEF. Note that the EDEI is generic. The SST index of poverty intensity is defined as either

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)x_i \tag{13}$$

or

$$I_{SST}(\mathbf{y}_p) = \frac{1}{n^2}\sum_{i=1}^{q}(2n - 2i + 1)x_i. \tag{14}$$

The C index $I_C(\overset{*}{\mathbf{y}})$ with the Gini EDEI $\Xi_G(\overset{*}{\mathbf{y}})$ is the SST index $I_{SST}(\overset{*}{\mathbf{y}})$; that is

$$I_{SST}(\overset{*}{\mathbf{y}}) = I^G_C(\overset{*}{\mathbf{y}}) = \frac{z - \Xi_G(\overset{*}{\mathbf{y}})}{z} = \Xi_G(\mathbf{x}) \tag{15}$$

[20] A function $f : \mathbf{R}^n_+ \to \mathbf{R}^1$ is S-concave if $f(\mathbf{By}) \geq f(\mathbf{y})$ for all $\mathbf{y} \in \mathbf{R}^n_+$, where $\mathbf{B}$ is a bistochastic matrix, a square matrix of order n with all elements being non-negative and the elements in each row/column being summed up to one (i.e., for elements b_{ij}'s of $\mathbf{B}$, $\sum_{j=1}^n b_{ij} = 1$ for $i = 1, 2, \ldots, n$; $\sum_{i=1}^n b_{ij} = 1$ for $j = 1, 2, \ldots, n$).

[21] It should be noted that Sen (1976) started from Axioms R (Ordinal Rank Weights), M (Monotonic Welfare), and N (Normalized Poverty Value) which have Gini social welfare implications.

[see Chakravarty (1997)]. Equation (15) provides a mathematical structure based on which one can see why the SST index is explicitly related to the Gini SEF.

From the above discussion, it becomes clear that a higher (lower) value of the poverty intensity measured by either I_S or I_{SST} means that the lower (higher) level of social welfare measured by the Gini SEF.

3.2 Common Multiplicative Decomposition

Although both S index and SST index do not permit additive decomposition, they permit a common multiplicative decomposition into the poverty rate, average poverty gap ratio and one plus the Gini index of poverty gap ratios.

Proposition 1 *The S index has the following multiplicative decomposition:*

$$I_S(\mathbf{y}_p) = H\overline{\mathbf{x}}_p \left(1 + G\left(\widetilde{\mathbf{x}}_p\right)\right), \qquad (16)$$

where $\widetilde{\mathbf{x}}_p$ has elements in non-decreasing order.

Proof: Using the relationship between G and Ξ_G [equation (6)] and equation (8) rewrite equation (11) as

$$I_S(\mathbf{y}_p) = H\Xi_G(\mathbf{x}_p) = H\overline{\mathbf{x}}_p \left(1 - G\left(\mathbf{x}_p\right)\right) = H\overline{\mathbf{x}}_p \left(1 + G\left(\widetilde{\mathbf{x}}_p\right)\right), \qquad (17)$$

where $\mathbf{x}_p$ ($\widetilde{\mathbf{x}}_p$) has elements in non-increasing (non-decreasing) order. □

As can be seen from the above proposition, we can view the S index as the product of the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios of the *poor*.

It is also interesting to compare the multiplicative decomposition of the original S index with the one presented here. As Sen (1976) pointed out, for a large q, the S index is defined as in equation (10) where the Gini index is for incomes of the poor. We show that the S index can be written alternatively as in equation (16) where the Gini index is for poverty gap ratios of the poor. Equation (16) is a bit simpler than equation (10) and permits a simpler geometric interpretation as shown later in this paper.

The following proposition states that the SST index permits similar multiplicative decomposition.

Proposition 2 *The SST index has the following multiplicative decomposition:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p \left(1 + G\left(\widetilde{\mathbf{x}}\right)\right), \tag{18}$$

where $\widetilde{\mathbf{x}}$ *has elements in non-decreasing order.*

Proof: Using the relationship between G and Ξ_G [equation (6)], equation (8) and $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$ rewrite equation (15) as

$$I_{SST}(\overset{*}{\mathbf{y}}) = \Xi_G(\mathbf{x}) = \overline{\mathbf{x}}\left(1 - G\left(\mathbf{x}\right)\right) = H\overline{\mathbf{x}}_p \left(1 + G\left(\widetilde{\mathbf{x}}\right)\right), \tag{19}$$

where $\mathbf{x}$ ($\widetilde{\mathbf{x}}$) has elements in non-increasing (non-decreasing) order. □

As can be seen from the above proposition, we can express the SST index as the product of the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios of the *population*.

Note that the two indices differ only by the argument of $G(\cdot)$. The S index has a component $G\left(\widetilde{\mathbf{x}}_p\right)$ while the SST index has a component $G\left(\widetilde{\mathbf{x}}\right)$. Since the poverty gap ratios of the non-poor sub-population are zeros and the poor and the non-poor sub-populations do not overlap in the censored income vector $\overset{*}{\mathbf{y}}$, the Gini index of poverty gap ratios of the *population* can be decomposed into two components as follows:

Lemma 1 *The Gini index of poverty gap ratios of the population,* $G\left(\widetilde{\mathbf{x}}\right)$*, is the sum of the Gini index of the average poverty gap ratios between the non-poor and the poor sub-populations,*$(1 - H)$*, and the poverty-rate-weighted Gini index of poverty gap ratio of the poor,* $HG\left(\widetilde{\mathbf{x}}_p\right)$*,as follows:*

$$G\left(\widetilde{\mathbf{x}}\right) = (1 - H) + HG\left(\widetilde{\mathbf{x}}_p\right). \tag{20}$$

Proof: From equation (6), we have

$$G\left(\mathbf{x}\right) = 1 - \frac{1}{n^2\overline{\mathbf{x}}} \sum_{i=1}^{n} \left(2n - 2i + 1\right) x_i, \tag{21}$$

where the poverty gap ratio vector of the population, $\mathbf{x}$ has elements in non-increasing order (i.e., the poor sub-population takes the top partition of the column vector while the non-poor sub-population takes the bottom partition). Similarly, from equation (6), we have

$$G\left(\mathbf{x}_p\right) = 1 - \frac{1}{q^2\overline{\mathbf{x}}_p} \sum_{i=1}^{q} \left(2q - 2i + 1\right) x_i, \tag{22}$$

where the poverty gap ratio vector of the poor, $\mathbf{x}_p$, has elements in non-increasing order. It is known that $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$. From equation (21) we get

$$G(\mathbf{x}) = 1 - \frac{q}{n}\frac{1}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)\,x_i - 2\left(1-\frac{q}{n}\right). \tag{23}$$

It can be further rewritten as

$$G(\mathbf{x}) = \frac{q}{n}\left\{1 - \frac{1}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)\,x_i\right\} - \left(1-\frac{q}{n}\right). \tag{24}$$

Thus,

$$G(\mathbf{x}) = (H-1) + HG(\mathbf{x}_p). \tag{25}$$

Applying equation (8) to G on the left-hand-side of equation (25), equation (25) becomes

$$G(\tilde{\mathbf{x}}) = -(H-1) - HG(\mathbf{x}_p). \tag{26}$$

Applying equation (8) to G on the right-hand-side of equation (26), equation (26) becomes

$$G(\tilde{\mathbf{x}}) = (1-H) + HG(\tilde{\mathbf{x}}_p). \tag{27}$$

□

Proposition 3 *The SST index and the S index are related in the following way:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = HI_S(\mathbf{y}_p) + 2H(1-H)\,\overline{\mathbf{x}}_p. \tag{28}$$

Proof: Combining the results in Lemma 1 and Proposition 2 gives

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p\left(2(1-H) + H(1+G(\tilde{\mathbf{x}}_p))\right). \tag{29}$$

Further manipulation of equation (29) gives equation (28). Zheng (1997) stated the same result [equation (3.9), p. 146] without giving the details of the proof. □

According to Chakravarty (1990, Theorem 6.9), if the SEF is completely strictly recursive, then

$$I_{BD}(\mathbf{y}_p) < I_C(\overset{*}{\mathbf{y}}). \tag{30}$$

In other words, the BD index is bounded above by the C index. For the Gini SEF, which is the underlying SEF for the S and SST indices but not completely strict recursive, the similar relationship holds.

Proposition 4 *The S index is bounded above by the SST index, i.e.,*

$$I_S(\mathbf{y}_p) < I_{SST}(\overset{*}{\mathbf{y}}). \tag{31}$$

Proof: Based on equation (28), we have

$$I_{SST}(\overset{*}{\mathbf{y}}) = HI_S(\mathbf{y}_p) + 2H(1-H)\overline{\mathbf{x}}_p$$

Given $2H(1-H)\overline{\mathbf{x}}_p > 0$ and $H < 1$,

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) &< HI_S(\mathbf{y}_p) \\ &< I_S(\mathbf{y}_p). \end{aligned}$$

□

The common multiplicative decomposability of the S and SST indices allows economists to evaluate social welfare measured by the poverty *intensity* and its contributing components. The multiplicative decomposition of the S and SST indices can be transformed, through the logarithmic transformation, to be additive in a simple form. In the following corollary, we use I for either the S index or the SST index and G for the Gini index of poverty gap ratios of either the poor or the population.

Corollary 1 *Since the S and SST indices of poverty intensity take the form of*

$$I = H\overline{\mathbf{x}}_p(1+G), \tag{32}$$

then

$$\Delta I = \Delta H + \Delta\overline{\mathbf{x}}_p + \Delta(1+G), \tag{33}$$

where $\Delta x = \ln x_t - \ln x_{t-1} \approx \frac{x_t - x_{t-1}}{x_{t-1}}$ *approximates the percentage change in* x *for a small change in* x.

Depending on the purpose of research, one may use the same poverty line $z_t = z_{t-1} = z$ for I_t and I_{t-1} and their components or different poverty lines z_t and z_{t-1}, respectively, for I_t and I_{t-1} and their components.

The common multiplicative decomposition also allows policy makers to use three specific anti-poverty policy "targets" (rate, gap, and inequality) in reducing poverty intensity. These targets may be used to monitor the effectiveness of the anti-poverty policy.[22]

[22]Bourguignon and Fields (1997) and Ravallion, van der Walle and Gautam (1995) discussed the relationship between poverty measures and anti-poverty policy actions. As

3.3 Similar Geometric Interpretations

The S index permits a simpler geometric interpretation that is somewhat different from that of Sen (1976) but is quite close to that of the SST index proposed by Shorrocks (1995). For comparison purpose, we also present and interpret the SST index geometrically

Note again that the relative deprivation meaure is $x_i = \frac{z-y_i}{z}$ if $z > y_i$, $x_i = 0$ otherwise, for $i = 1, 2, \ldots, n$. The x_i's are in non-increasing order. The first q x_i's are positive for the poor who are deprived and the rest are zeros for the non-poor.

The deprivation profile can be graphed by plotting $\frac{1}{n}\sum_{i=1}^{r} x_i$ against $\frac{r}{n}$ for $r = 1, 2, \ldots, n$ in a unit box. As shown in Figure 1, the poverty profile starts from the origin, reaches out concavely to the point a and then becomes horizontal from the point a to the point $H\overline{\mathbf{x}}_p$. The point H represents the poverty rate, and the point $H\overline{\mathbf{x}}_p$ represents the average poverty gap ratio of the population, $\overline{\mathbf{x}}$. Since the deprivation measures $\{x_i\}$ are in non-increasing order, the concave arc $0a$ is in fact an inverted generalized Lorenz curve for the deprivation measures $\{x_i\}$, which represents the inequality of poverty gap ratios of the poor. The dotted straight line linking the origin 0 and the point a would be a segment of the poverty profile if the poor had identical incomes (i.e., their poverty gap ratios were all identical). Since the non-poor have zero deprivation, the horizontal segment from the point a to the point $H\overline{\mathbf{x}}_p$ of the deprivation profile has no significant information but shows the non-poor account for the $1 - H$ proportion of the population.

[Insert Figure 1 about here]

In Figure 2, we show the S index has a simple geometric interpretation that is similar to that of the Gini index. Note that triangle $0H'H$ is area E. Triangle $0Ha$ is area C. The space between arc $0a$ and the dotted straight

Bourguignon and Fields (1997) noted, if there is a qualitative difference (e.g. in functions) between being poor or non-poor, the poverty rate is of specific interest. Similarly, if the depth of poverty is of a major social concern, the average poverty gap ratio is of specific interest. If the dispersion of deprivations demands more social attention, inequality of deprivations is clearly of greater importance. But in practice the changes in inequality of deprivations over time or across jurisdictions, relative to those in the poverty rate or average poverty gap ratio, are of much smaller magnitude. See Osberg and Xu (1997, 2000).

line linking the origin 0 and the point a is area D. Thus,

$$\text{Area } E = \frac{1}{2}H. \tag{34}$$

$$\text{Area } C = \frac{1}{2}H^2\overline{\mathbf{x}}_p. \tag{35}$$

Area D can be computed from the fact that the Gini index of poverty gap ratios of the poor is given by[23]

$$G(\widetilde{\mathbf{x}}_p) = \frac{\text{Area } D}{\text{Area } C'} = \frac{\text{Area } D}{\text{Area } C}. \tag{36}$$

Using equations (35) and (36) yields

$$\text{Area } D = \text{Area } C \times G(\widetilde{\mathbf{x}}_p) = \frac{1}{2}H^2\overline{\mathbf{x}}_p G(\widetilde{\mathbf{x}}_p)$$

The S index is simply the ratio of the sum of areas C and D to area E, i.e.,

$$\begin{aligned} I_S(\mathbf{y}_p) &= \frac{\text{Area } C + \text{ Area } D}{\text{Area } E} \\ &= \frac{\frac{1}{2}H^2\overline{\mathbf{x}}_p + \frac{1}{2}H^2\overline{\mathbf{x}}_p G(\widetilde{\mathbf{x}}_p)}{\frac{1}{2}H} \\ &= H\overline{\mathbf{x}}_p(1 + G(\widetilde{\mathbf{x}}_p)). \end{aligned} \tag{37}$$

[Insert Figure 2 about here]

For a better understanding of the common multiplicative decomposition and similar geometric interpretations, we also analyze the geometric interpretation of the SST index in a similar fashion in Figure 3. Let the lower triangle of the unit box in Figure 3 be area A and the rectangle at the lower right-hand corner of the unit box be area B. Thus

$$\text{Area } A = \frac{1}{2} \tag{38}$$

[23] Note that area C', the triangle formed by two dotted straight lines and the vertical axis, is identical to area C. Also note that the Gini index of the poor's poverty gap ratios ($\mathbf{x}$) here is defined as the ratio of two areas in the rectangle of the length H and the height $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$ in the larger unit box. This rectangle can be transformed into a unit box by rescaling the vertical and horizontal axes without affecting the ratio of the two areas and hence the Gini index of the poor's poverty gap ratios is defined on the ratio of the two areas.

and

$$\text{Area } B = (1-H)H\overline{\mathbf{x}}_p = H\overline{\mathbf{x}}_p - H^2\overline{\mathbf{x}}_p. \quad (39)$$

According to equation (18), the SST index can be expressed as

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p(1 + G(\widetilde{\mathbf{x}})). \quad (40)$$

Further, using equations (20), equation (40) becomes

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p(2 - H + HG(\widetilde{\mathbf{x}}_p)). \quad (41)$$

Now compute the ratio of the sum of areas B, C, and D to area A, i.e.,

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) &= \frac{\text{Area } B + \text{Area } C + \text{Area } D}{\text{Area } A} \\ &= \frac{H\overline{\mathbf{x}}_p\left[(1-H) + \frac{1}{2}H + \frac{1}{2}HG(\widetilde{\mathbf{x}}_p)\right]}{\frac{1}{2}} \\ &= H\overline{\mathbf{x}}_p(2 - H + HG(\widetilde{\mathbf{x}}_p)) \end{aligned} \quad (42)$$

Thus, the SST index is the ratio of the sum of areas B, C and D to area A.

[Insert Figure 3 about here]

The similar geometric interpretation puts both S and SST indices in a Gini-like framework which shows clearly that H, $\overline{\mathbf{x}}_p$, and G are three key components determining the poverty intensity. For applied economists and policy analysts, this graphical approach can convey information about poverty effectively.

4 An Illustrative Example

As an illustrative example, we apply the SST index and its decomposition to analyze the trend of poverty among working-age households with head less than 65 years of age in the United Kingdom from 1974 to 1995. Luxembourg Income Study (LIS) data for 1974, 1979, 1986, 1991, and 1995 are used.[24] Poverty intensity is measured by the SST index.

[24]For 1974, 1979, 1986, 1991, and 1995, the sample size of the survey is 6695, 6777, 7178, 7056, and 6797, respectively. The total household after-tax income is used to compute the individual equivalent income in each household according to the LIS scale; that is, the individual's equivalent income is defined as the household after-tax money income divided by the square root of the number of the members of the household. The poverty line in a year is estimated by 1/2 the median of the distribution of individual equivalent incomes in that year.

As already discussed, a higher value of the SST index corresponds to a lower level of the social welfare due to poverty and the SST index can be viewed as the product of three measures: the poverty rate, average poverty gap ratio, and one plus the Gini index of poverty gap ratios of the population.[25] These three measures represent the incidence, depth, and distribution of poverty, respectively. A higher value of each measure corresponds to a lower level of social welfare due to poverty, if the other two components of poverty intensity remain unchanged.

While poverty intensity provides an overall measure of social welfare due to poverty, its decomposition into three components allows analysts to better understand poverty. Most importantly, poverty intensity may change in a different direction to changes in each contributing component over time. In Panel C of Table 1, for example, one can see that between 1974 and 1979 the poverty rate among workless households fell (from 66 % to 52.8 %), at the same time as the intensity of poverty rose (from 22.5 % to 27.3 %). The reason is that there was a large increase in the poverty gap (from 21.3 % to 30.4 %). Over all, as Panel A indicates, the poverty intensity of all working-age households fell from 2.3% in 1974 to 1.5% in 1979 showing an improvement in social welfare. This resulted from a sharp fall of the poverty rate from 5.8% in 1974 to 3.1 % in 1979, which outweighed a slight rise of the poverty gap[26] from 19.9 % in 1974 to 23.6 % in 1979. [There was only a very small change in inequality of the poor (measured as one plus the Gini index)]. However, the key point is that if its underlying components move in different directions, trends in poverty intensity cannot necessarily be derived from analysis of a single component.

From 1979 to 1986, poverty intensity increased substantially, from 1.5 % to 5.2 %. Worsening poverty intensity in 1986 was a result of an increase in *both* the poverty rate and poverty gap.[27] The change in poverty intensity from 1986 (5.2 %) to 1991 (6.4 %) resulted from a higher poverty rate (8.7 % became 12.8 %) outweighing the impact of a lower poverty gap (30.4 %

[25]Each variable can be expressed as either a ratio or a percentage, but neither convention is essential. We present them in a way that is consistent with convention for easy communication of research results.

[26]For simplicity, the average poverty gap ratio is called the poverty gap.

[27]Since the change in inequality was very small ($(1.958 - 1.982)/1.982 = -0.012$), it was dominated by changes in the rate and gap. Indeed, since it is generally true we henceforth omit the direct discussion of changes in $1 + G$.

became 25.7 %).

[Insert Table 1 about here]

To help readers visualize trends in poverty intensity, and its main components, Figure 4 presents five "poverty boxes," one for each year, for working-age households in the United Kingdom. The poverty box is a rectangle whose base is the poverty rate and whose height is the poverty gap. All poverty boxes are aligned at the origin. The poverty box representation enables the reader to see quickly the impact of changes in the poverty rate and poverty gap on all working-age households over the years. As the $1 + G$ column of Table 1 indicates, the amount of inequality among the poor is fairly constant over time,[28] so changes in the SST index over time can be graphically represented by changes in the size of the poverty box. Since humans generally do better in extracting relative size information from graphs than from arrays of numbers, graphing poverty boxes over time is a useful way of presenting information on the change in poverty intensity.

[Insert Figure 4 about here]

In the particular case of UK poverty trends, Figure 4, which is based on the data in Panel A of Table 1, is fairly clear in showing that poverty intensity fell from 1974 to 1979 and rose from 1979 onwards. Figure 4 also shows the main contributing factors of the change in poverty intensity. For example, the fall of poverty intensity from 1974 to 1979 resulted from a sharply decreased poverty rate which outweighed the increased poverty gap. From 1986 to 1991, the rise of poverty intensity was caused by an increased

[28] As Osberg (2000, p. 852, footnote 8 and references therein) noted, across LIS countries and over years the coefficient of variation of poverty rates is 0.493, and for average poverty gap ratios it is 0.185. However, the coefficient of variation of $1 + G$ is only 0.014. He also noted that, for Canadian provinces and US states in 1997, the coefficient of variation is 0.341 for the SST index, 0.384 for the poverty rate, 0.141 for the poverty gap ratio and 0.011 for $1 + G$. The "common sense" explanation for this phenomenon is that the differences in incomes among the poor are small when compared to income differences among the non-poor sub-population. The upper bound on the incomes of poor people is the poverty line. The lower bound (leaving aside measurement error) is subsistence. The dollar value of the difference is unlikely to be very large, particularly when compared to the dollar differences among the non-poor sub-population.

poverty rate which dominated the decreased poverty gap. In 1995, poverty intensity rose because both the poverty rate and poverty gap rose.

One of the most important policy issues in poverty analysis is employment. Whatever the reason why they have no work, those working-age households without earnings are typically the poorest of the poor. As demonstrated by Panels B and C of Table 1, a substantial rise in the proportion of workless households has been responsible for much of the upward trend in poverty intensity among the British working-age households. Back in the days of the Welfare State under Old Labor in 1974, 95.3% of working-age households had some work. A lasting legacy of the Thatcher years seems to have been a very substantial increase in the proportion of workless households — the big jump (from 7.5% to 17.2%) between 1979 and 1986 has not been followed by a lasting decline (15.6 %) in 1991 and over the period from 1974 to 1995 worklessness rose from 4.7% to 20.4% — i.e. by 15.7 percentage points.

Panels B and C of Table 1 show clearly that the working condition of households affects their ultimate social welfare due to poverty. Poverty intensity for workless households can be as high as 27.3 % (in 1979) while the highest level of poverty intensity for the working households was only 2.6 % (in 1991 and 1995).

When we examine working and workless households (see Panels B and C of Table 1), we note that the poverty rate fell in 1995 for both working households (from 4.9 % in 1991 to 4.6 % in 1995) and workless households (from 55.8 % in 1991 to 47.4 % in 1995) households. If one were to use only the poverty rate to measure poverty, this would naturally lead to the conclusion that poverty was reduced in 1995 relative to that in 1991. However, within each sub-population poverty intensity was constant. Overall poverty intensity increased as the proportion of workless households rose. As discussed earlier in this paper, the poverty rate would indeed mislead policy makers in this case.

If we use the SST index and its decomposition, we find that in fact poverty intensity did not change from 1991 to 1995 for both types of households: 2.6 % for the working households and 23.3 % for the workless households in both 1991 and 1995. For working households, the poverty gap rose from 27.2 % in 1991 to 28.8 % in 1995 while for workless households, the poverty gap increased to 27.8 % in 1995 (from 25 % in 1991). Within each sub-population, the change in the poverty gap offset the decline in the poverty rate. However, as Panel A indicates, there was an overall increase in poverty among all working-age households, due to the rising proportion of workless households.

The information in Panels B and C of Table 1 can also be presented using the poverty box. Aggregate poverty intensity can be roughly viewed as the weighted average of poverty intensity in sub-populations with the weights being relative sub-population sizes. Generally, workless households are only a small fraction of working-age households. However, because employment is generally the main source of income, working poor households are a much smaller fraction of working households and workless poor households represent a larger fraction of workless households. Since the workless and working sub-populations differ so much in the various aspects of poverty, unbundling the aggregate poverty intensity of the working-age households into poverty intensity and its main dimensions across different sub-populations is crucial.

Figures 5 and 6 are drawn to illustrate how to use the poverty box to effectively present information on poverty intensity and its dimensions across two sub-populations over time. In these two figures, the horizontal axis represents 100 % of the population of our study. It is demarcated by a vertical line with the left segment representing the percentage of workless households and the right segment representing the percentage of working households.[29] The vertical axis measures the poverty gap. For each sub-population (working and workless households) a poverty box is drawn at the lower left corner of each demarcated rectangle with the poverty gap as the height and the poverty rate as the base.[30] As before, for each sub-population, poverty intensity can be represented by the size of the area in the poverty box.

[Insert Figures 5 and 6 about here]

A comparison of Figures 5 and 6 shows that the rise in aggregate poverty intensity from 1974 to 1995 is attributable to (1) changes of the poverty boxes of two sub-populations over time —an increase in the poverty gap for

[29]The horizontal axis is normally truncated at 50 % because no extra information is presented beyond this point.

[30]Since the poverty rate of a sub-population is presented along the horizontal axis for the population, it must be translated into the poverty rate of the the population observed in the sub-population by multiplying the poverty rate of the sub-population by the percentage of the sub-population in the population.

both the working and workless households, a decrease in the poverty rate for the workless households, and an increase in the poverty rate for the working households; and (2) an increase in the percentage of workless households.

The 1974 data in Figure 5 also illustrate the importance of the relative size of sub-populations, when considering the poverty rate. In looking at Table 1 without factoring in the relative size of sub-populations, one is likely to miss the fact that the percentage of working poor households in the population is 2.7 % (= 2.8 % × 95.3 %), only slightly lower than that of workless poor households in the population (3.1 % = 66.0 % × 4.7 %).

Figures 5 and 6 also illustrate the role which differing level of worklessness play in determining aggregate poverty. Although the poverty rate of workless households actually fell from 66.0 % in 1974 to 47.4 % in 1995, the percentage of workless population rose so dramatically (from 4.7 % in 1974 to 20.4 % in 1995) that the percentage of workless poor households in the population rose from 3.10 % (66.0 % × 4.7 % = 3.1 %) in 1974 to 9.7 % (47.4 % × 20.4 % = 9.7 %) in 1995.

As this example indicates, the multiplicative decomposition of the Sen family of poverty indices and simple geometric interpretations is useful for policy analysis. More importantly, given the small variation of $1 + G$ over time for either the population or each of sub-population, comparative study using poverty boxes can serve as a very effective tool for communicating research results.

5 Concluding Remarks

This paper has discussed Sen's axiomatic approach to poverty measures, highlighted the common underlying social evaluation function for the Sen indices of poverty intensity, and presented a unified multiplicative decomposition framework for the Sen indices.

The Sen indices (the S and SST indices) share a common Gini social evaluation function and have a common multiplicative decomposition structure—being the product of the poverty rate, average poverty gap ratio of the poor and one plus the Gini index of the poverty gap ratios as follows:

$$\begin{pmatrix} \text{The S} \\ \text{index} \end{pmatrix} = \begin{pmatrix} \text{poverty} \\ \text{rate} \end{pmatrix} \times \begin{pmatrix} \text{average} \\ \text{poverty} \\ \text{gap} \\ \text{ratio} \end{pmatrix} \times \begin{pmatrix} 1 + \text{Gini index} \\ \text{of poverty gaps} \\ \text{of the poor} \end{pmatrix}$$

and

$$\begin{pmatrix} \text{The SST} \\ \text{index} \end{pmatrix} = \begin{pmatrix} \text{poverty} \\ \text{rate} \end{pmatrix} \times \begin{pmatrix} \text{average} \\ \text{poverty} \\ \text{gap} \\ \text{ratio} \end{pmatrix} \times \begin{pmatrix} 1 + \text{Gini index} \\ \text{of poverty gaps} \\ \text{of the population} \end{pmatrix}.$$

This common multiplicative decomposition structure (1) gives the two indices a much more straightforward interpretation of poverty intensity, (2) allows the indices to be computed much more easily via commonly known poverty measures (the poverty rate and average poverty gap ratio) and inequality measures (the Gini index of the poverty gap ratios), and (3) permits the indices to have the Gini-index-like geometric interpretations. The practical implication of the multiplicative decomposition is that the Sen indices can be linearized so that the percentage change in these indices are additively decomposable.

The results of this paper also show that the Sen poverty indices can be expressed in a simple graphical form and interpreted easily so that they can be accessible by, and useful to, policy makers in both developed and developing countries.

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Table 1: Poverty Trend: Households with Head Less Than 65 years of Age in the United Kingdom

Year	*Hhds* %	*Pov. Intensity* %	*Pov. Rate* %	*Pov. Gap* %	*1 + G*
		Panel A: All Working-Age Households			
1974	100.0	2.3	5.8	19.9	1.969
1979	100.0	1.5	3.1	23.6	1.982
1986	100.0	5.2	8.7	30.4	1.958
1991	100.0	6.4	12.8	25.7	1.929
1995	100.0	7.2	13.3	28.1	1.934
		Panel B: Working Households			
1974	95.3	1.0	2.8	18.4	1.988
1979	92.5	1.5	3.1	23.6	1.982
1986	82.8	2.5	4.1	31.0	1.978
1991	84.4	2.6	4.9	27.2	1.974
1995	79.6	2.6	4.6	28.8	1.976
		Panel C: Workless Households			
1974	4.7	22.5	66.0	21.3	1.603
1979	7.5	27.3	52.8	30.4	1.701
1986	17.2	17.3	31.0	30.1	1.862
1991	15.6	23.3	55.8	25.0	1.676
1995	20.4	23.3	47.4	27.8	1.768

Note: The Luxembourg Income Study (LIS) database for the United Kingdom is used. The individual's equivalent income is defined as the household after-tax money income divided by the square root of the number of the members of the household. The poverty line in a year is estimated by 1/2 the median of the distribution of individual equivalent incomes in that year. A working household has positive earnings from wages/salaries or self-employement. Poverty intensity = pov. rate $\times$ pov. gap $\times$ $(1 + G)$ where G represents the Gini index of poverty gap ratios or inequality of deprivations of the population.

Figure 1: Deprivation Profile

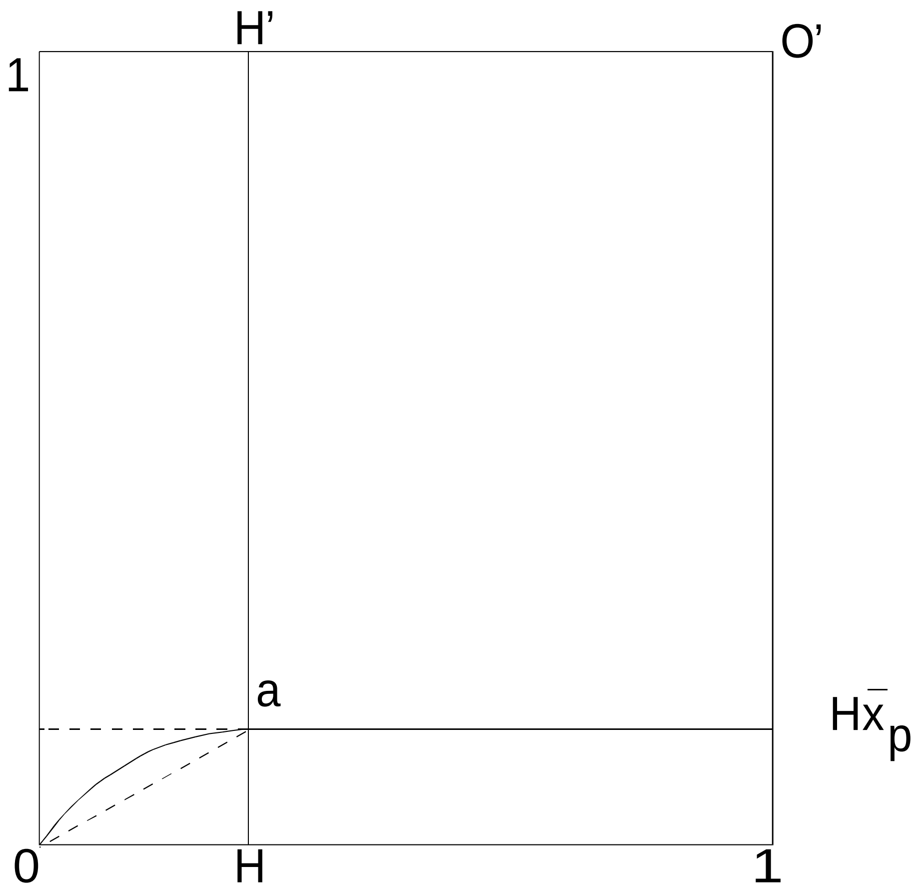

Figure 2: Geometric Interpretation of the S Index

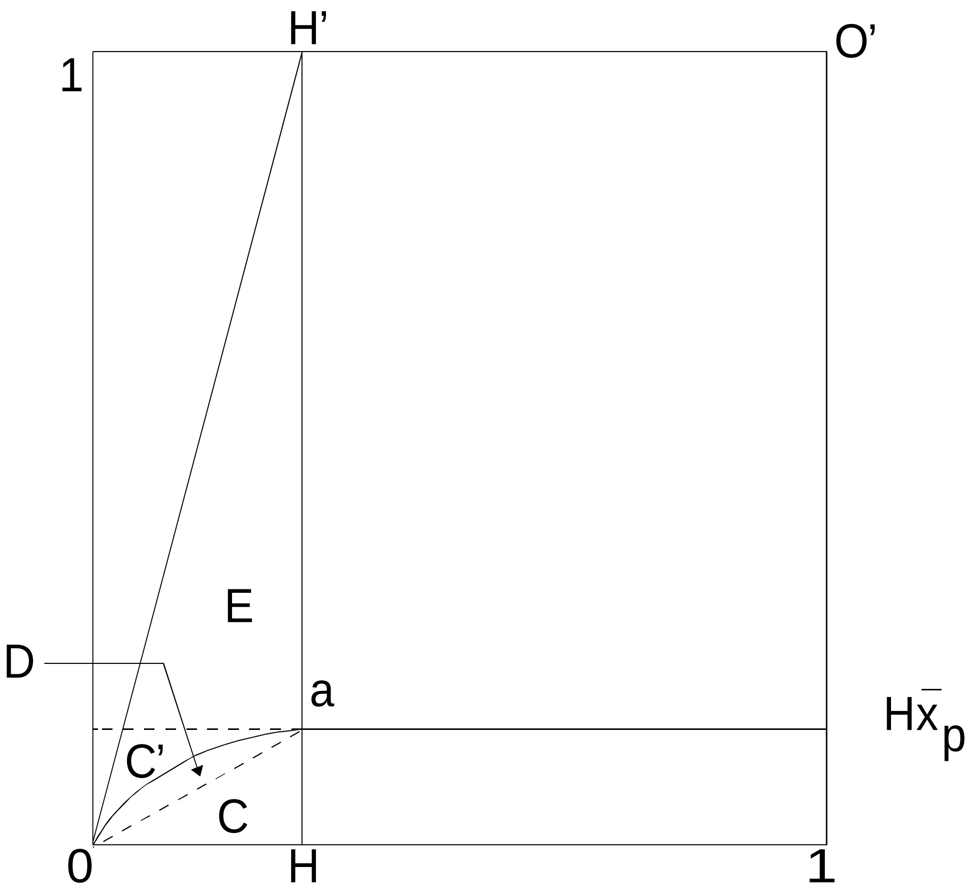

Figure 3: Geometric Interpretation of the SST Index

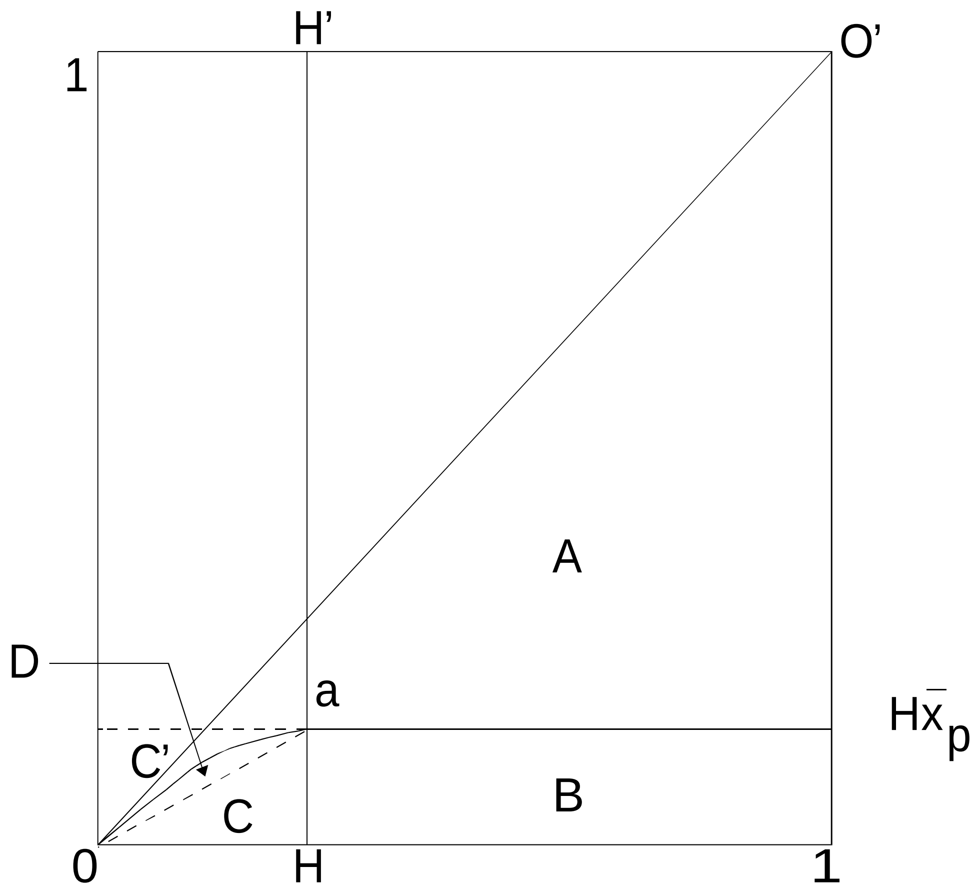

Figure 4: Poverty Box for UK Working-Age Households: 1974, 1979, 1986, 1991, and 1995

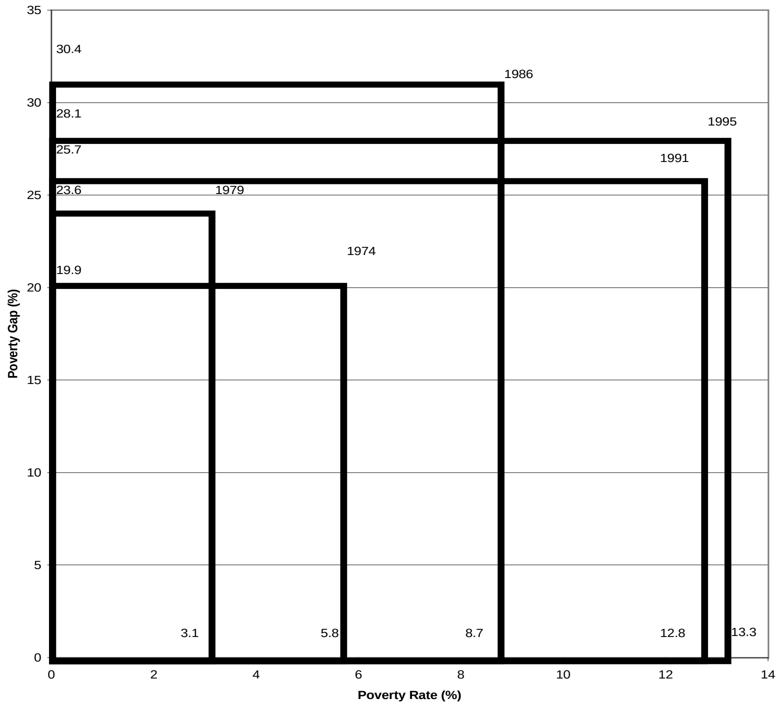

Figure 5: Poverty Box for UK Workless and Working Households: 1974

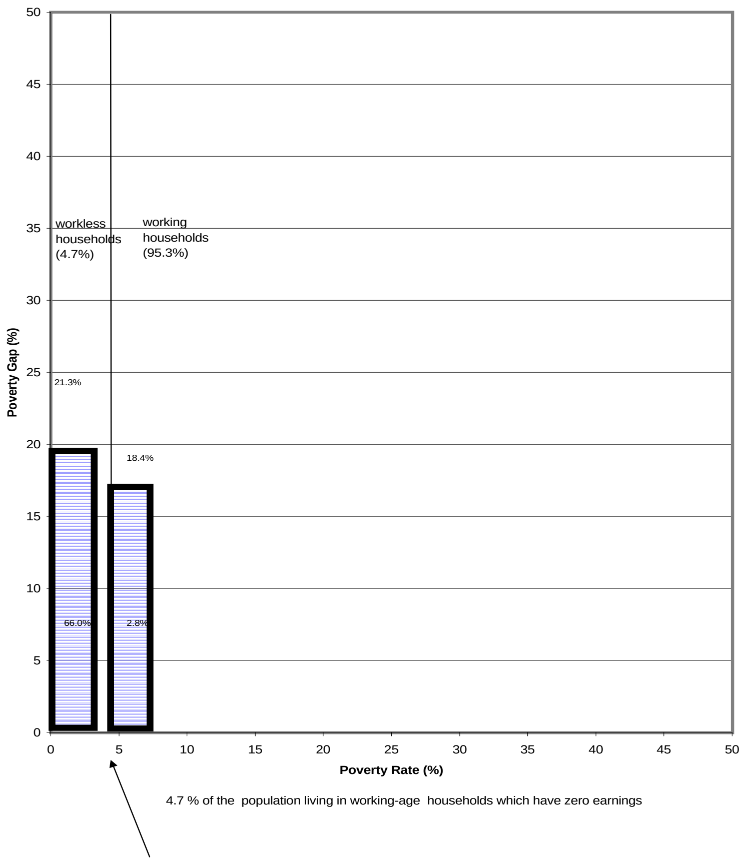

Figure 6: Poverty Box for UK Workless and Working Households: 1995